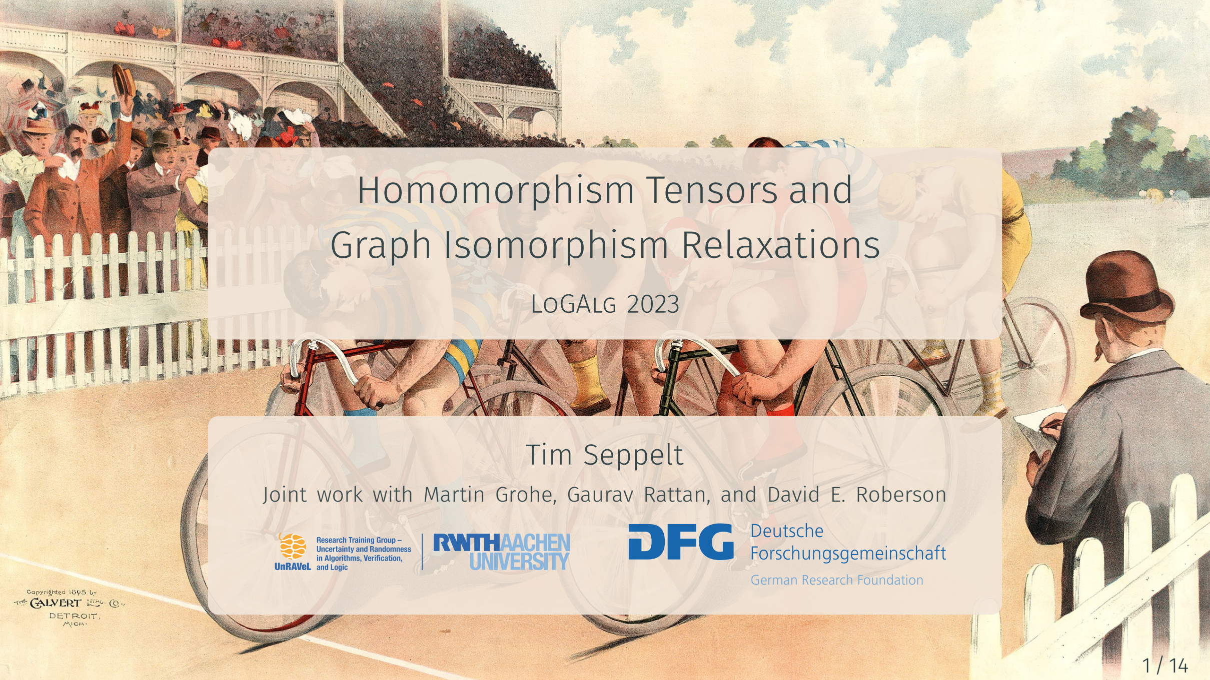

Homomorphism Tensors and Graph Isomorphism Relaxations

LoGAlg 2023

Tim Seppelt

Joint work with Martin Grohe, Gaurav Rattan, and David E. Roberson

UnRAVeL Research Training Group – Uncertainty and Randomness in Algorithms, Verification, and Logic

RWTH AACHEN UNIVERSITY

DFG Deutsche Forschungsgemeinschaft
German Research Foundation

G and H are isomorphic iff
integer program $\mathrm{ISO}(G, H)$ is feasible

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⋮	⋮
3	3
2	2
1	1
Lasserre	Sherali–Adams
semidefinite prog.	linear prog.

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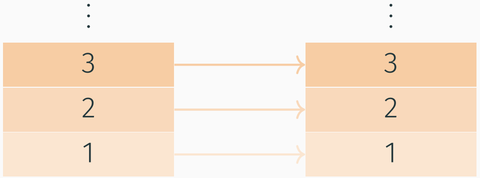

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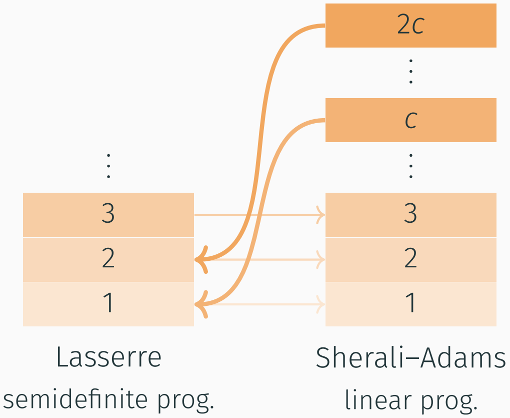

Atserias and Ochremiak (2018)

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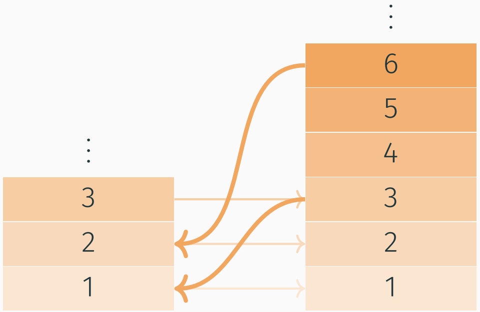

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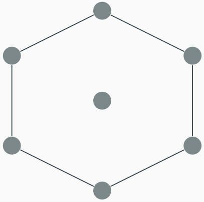
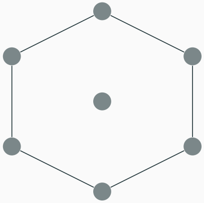
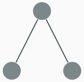
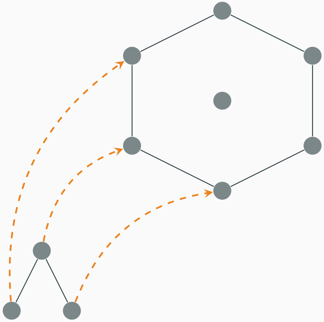
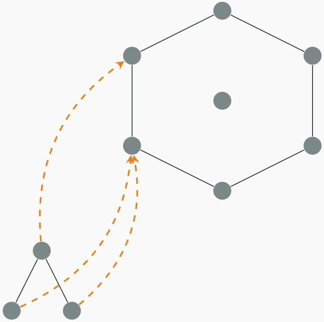
Atserias and Ochremiak (2018), **Roberson & S. (ICALP 2023)**

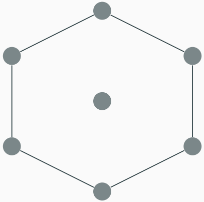


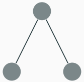












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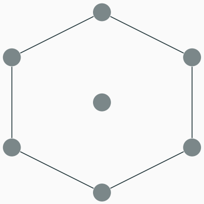

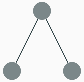

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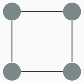

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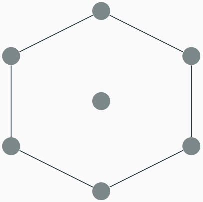
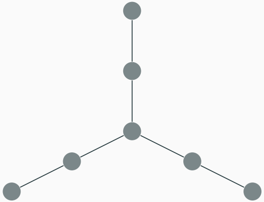
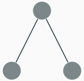
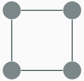
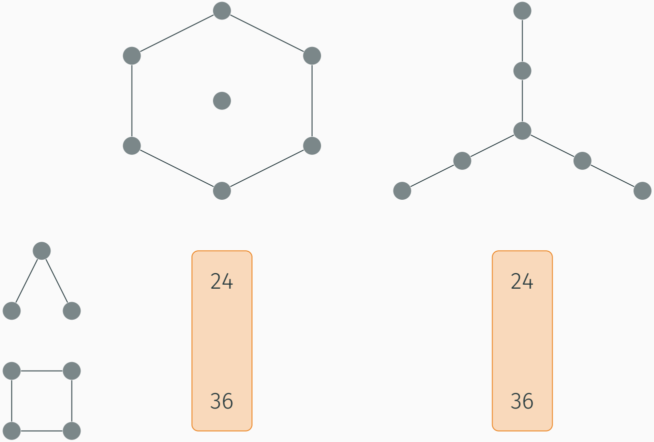
		
	24	24
	36	36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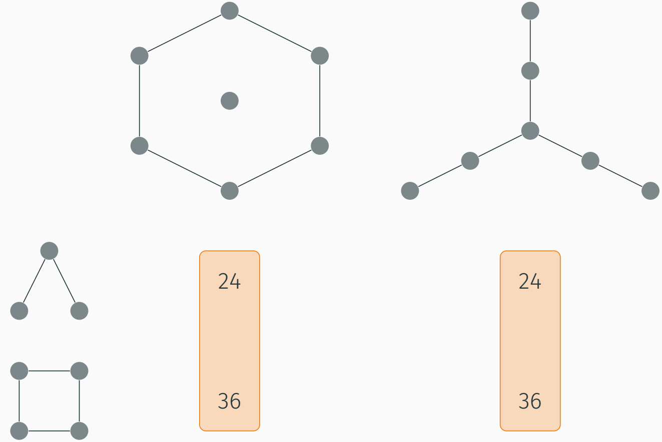
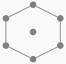
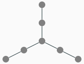


The graphs  and  are **homomorphism indistinguishable** over {, }.

Homomorphism Indistinguishability

graph class $\mathcal{F}$	relation $\equiv_{\mathcal{F}}$	
all graphs	isomorphism	Lovász (1967)

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...	...	

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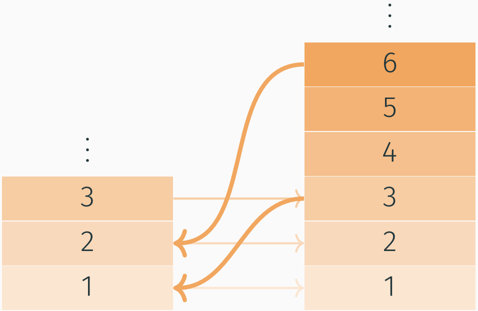

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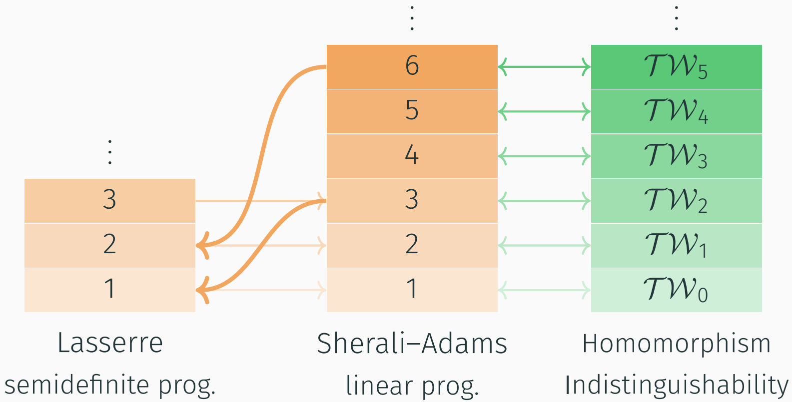

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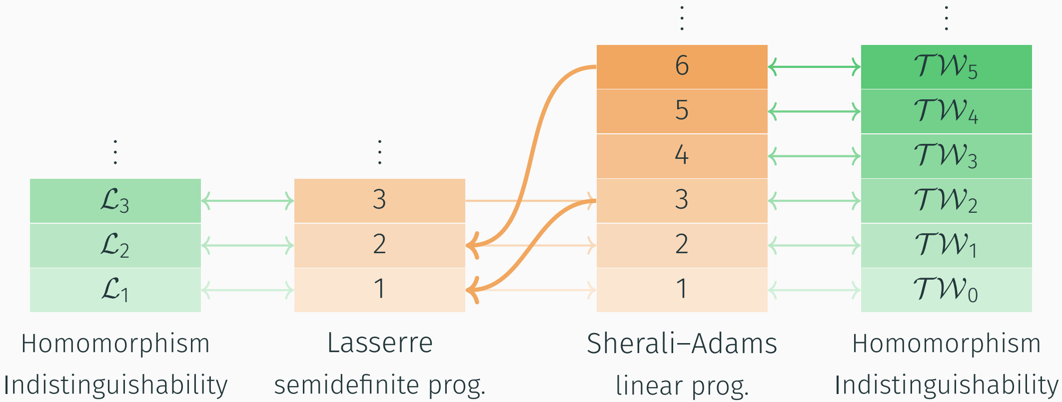

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homomorphism tensors,
algebraic operations,
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Graphs

(bi)labelled graphs,
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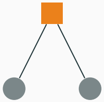
Labelled Graphs and Homomorphism Vectors



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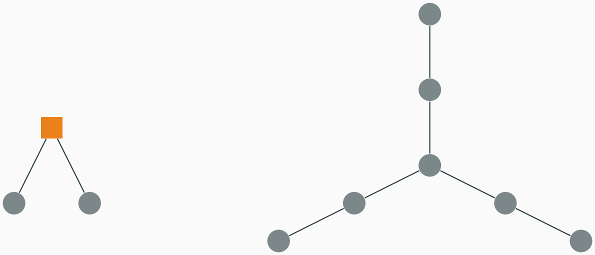

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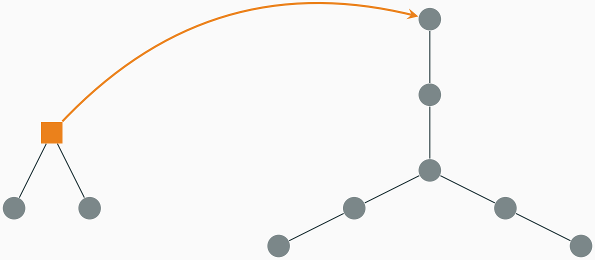

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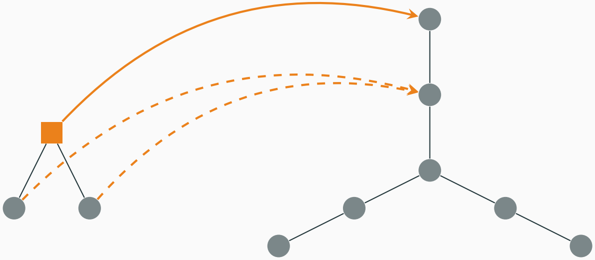

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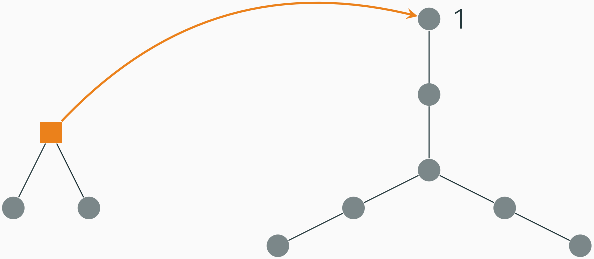

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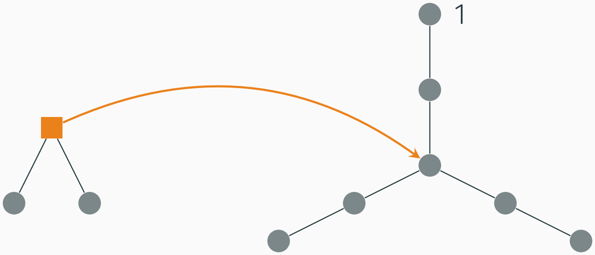

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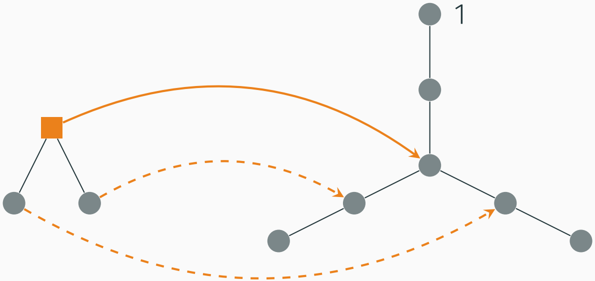

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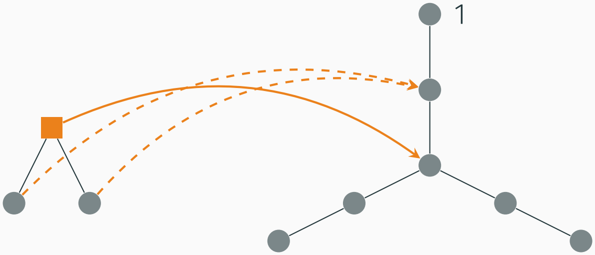

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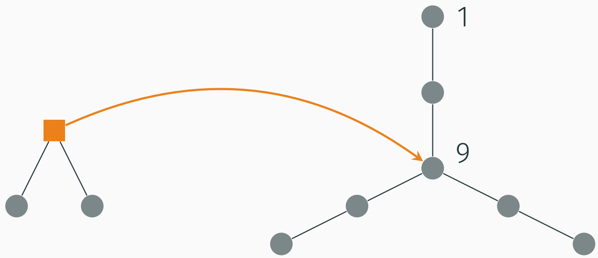
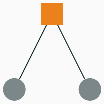
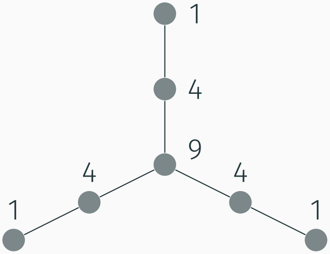


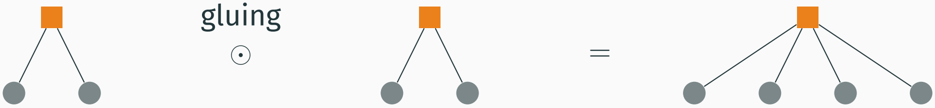
Labelled Graphs and Homomorphism Vectors

$$\mathcal{F} \longrightarrow \mathbb{C}^{V(G)}$$



$\mapsto$




gluing
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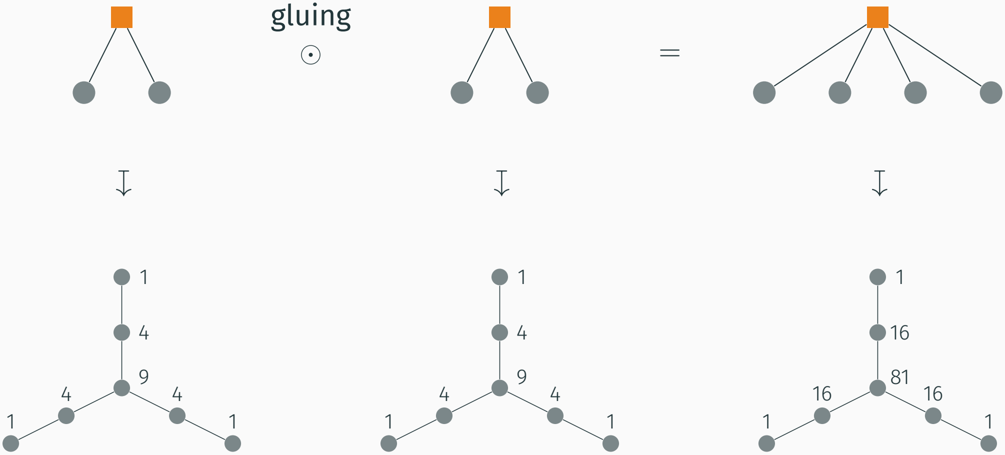
Combinatorial and Algebraic Operations



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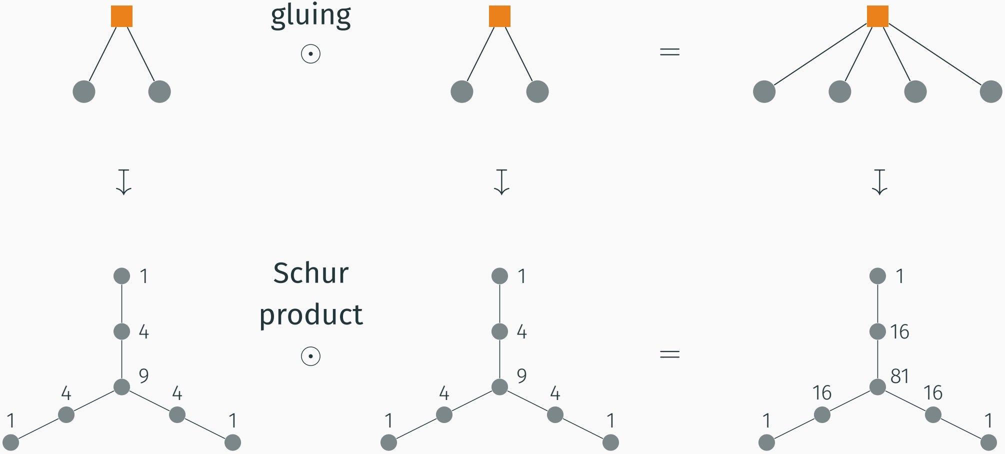

Simultaneous Similarity $\longleftrightarrow$ Homomorphism Indistinguishability

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Let $A_1, \dots, A_n$ and $B_1, \dots, B_n$ be complex square matrices.

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Theorem (Grohe, Rattan, S. (ICALP 2022))

X pseudo-stochastic s.t. $XA_i = B_i X$ $\Longleftrightarrow$ For every word $w \in [n]^*$, $\operatorname{soe} w_A = \operatorname{soe} w_B$.

Equations

homomorphism tensors,
algebraic operations,
simultaneous similarity

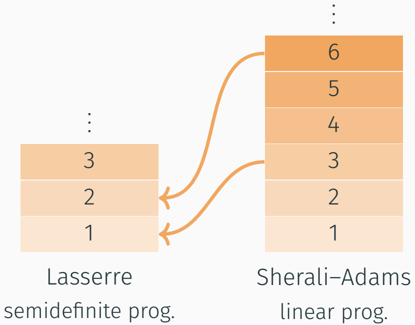
$\longleftrightarrow$

Graphs

(bi)labelled graphs,
combinatorial operations,
homomorphism indist.

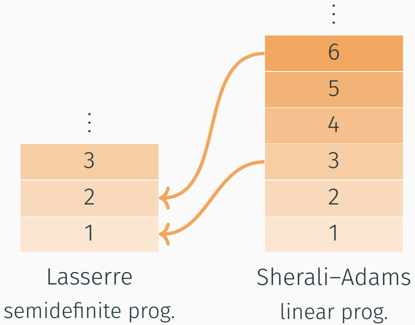
Upper Bound

- $\mathcal{L}_t \subseteq \mathcal{TW}_{3t-1}$,



Upper Bound

- $\mathcal{L}_t \subseteq \mathcal{TW}_{3t-1}$,
- $\mathcal{L}_t$ contains the clique K_{3t}.



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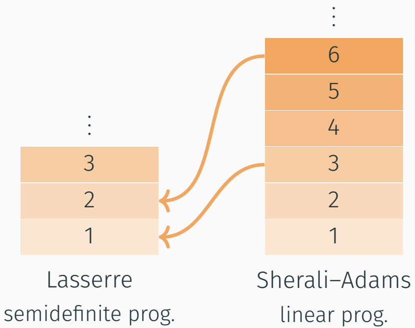

$\mathcal{L}_t$ is a class of graphs of treewidth $\leq 3t-1$ containing K_{3t}.

Lower Bound

$\mathcal{L}_t$ is a class of graphs of treewidth $\leq 3t - 1$ containing K_{3t}.

Although $\mathcal{L}_t \not\subseteq \mathcal{TW}_{3t-2}$, it could well be that $G \equiv_{\mathcal{TW}_{3t-2}} H \implies G \equiv_{\mathcal{L}_t} H$.

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Conjecture (Roberson (2022))

Every minor-closed union-closed graph class is homomorphism distinguishing closed.

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Corollary (Roberson and S. (ICALP 2023))

For every $t \geq 1$, there are graphs G and H such that $G \simeq^{\mathrm{SA}}_{3t-1} H$ and $G \not\simeq^{\mathrm{L}}_{t} H$.

G and H are isomorphic iff
integer program $\mathrm{ISO}(G, H)$ is feasible

$\vdots$

3

2

1

Lasserre
semidefinite prog.

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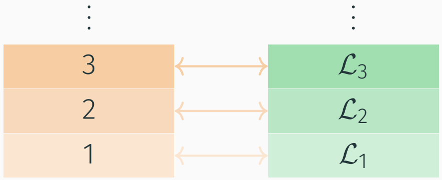

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Homomorphism
Indistinguishability

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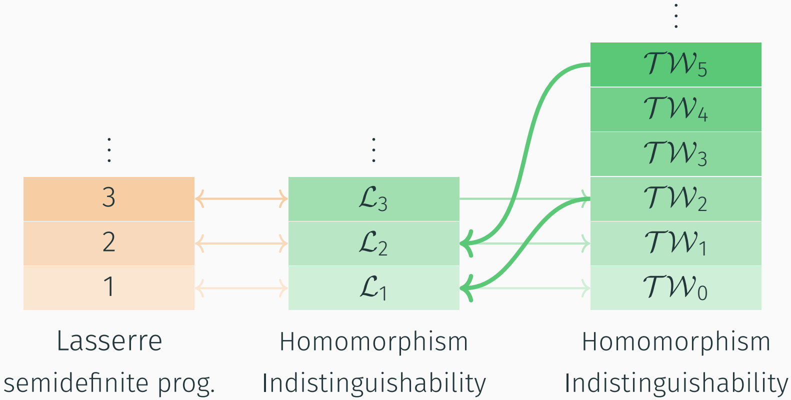

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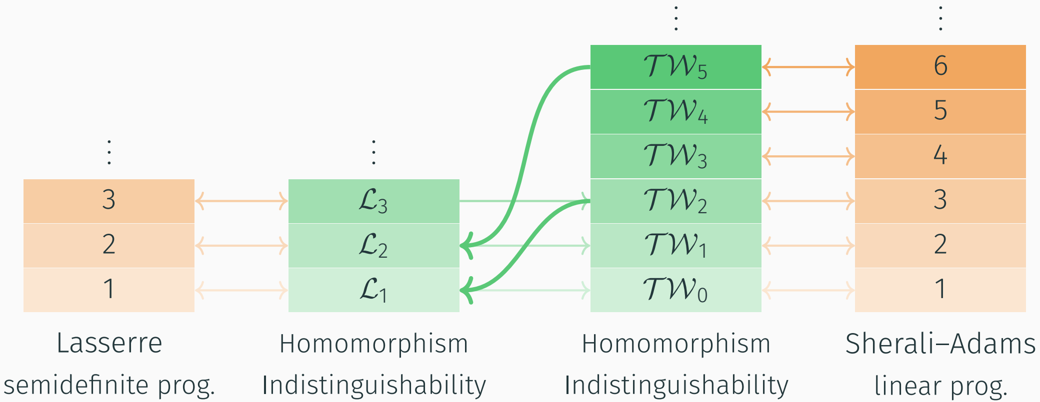
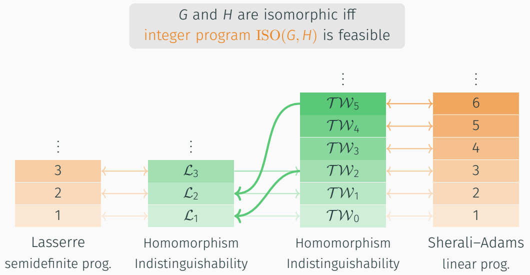
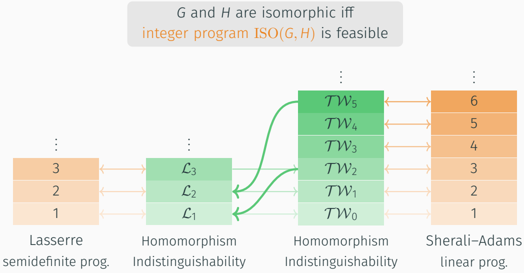


- Homomorphism indistinguishability characterisations of ISO relaxations



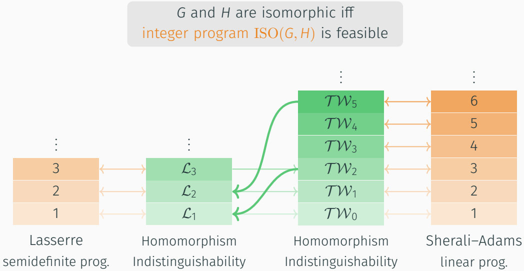
Conclusion

- Homomorphism indistinguishability characterisations of ISO relaxations
- Homomorphism tensors of (bi)labelled graphs



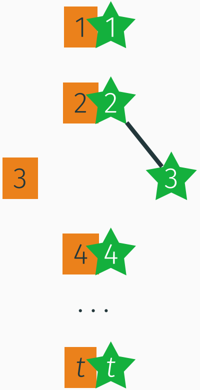
Conclusion

- Homomorphism indistinguishability characterisations of ISO relaxations
- Homomorphism tensors of (bi)labelled graphs
- Determined number of Sherali–Adams levels necessary to guarantee feasibility of Lasserre



The Graph Class $\mathcal{L}_t$

A (t, t)-bilabelled graph is *atomic* if all its vertices are labelled.



The Graph Class $\mathcal{L}_t$

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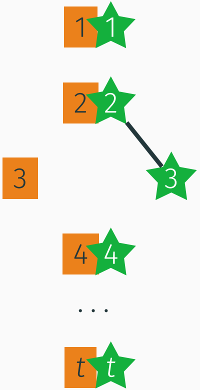
The class $\mathcal{L}_t$ is generated by atomic graphs under

- series composition,
- parallel composition with atomic graphs,
- permutation of labels.



Extras: Lasserre

Let $t \geq 1$. The *level-t Lasserre relaxation for graph isomorphism* has variables y_I ranging over $\mathbb{R}$ for $I \in \binom{V(G)\times V(H)}{\leq 2t}$. The constraints are

$$
\begin{aligned}
M_t(y) := (y_{I\cup J})_{I,J\in\binom{V(G)\times V(H)}{\leq t}} &\succeq 0,\\
\sum_{h\in V(H)} y_{I\cup\{gh\}} &= y_I \text{ for all } I \text{ s.t. } |I| \leq 2t-2 \text{ and all } g \in V(G),\\
\sum_{g\in V(G)} y_{I\cup\{gh\}} &= y_I \text{ for all } I \text{ s.t. } |I| \leq 2t-2 \text{ and all } h \in V(H),\\
y_I &= 0 \text{ if } I \text{ s.t. } |I| \leq 2t \text{ is not partial isomorphism}\\
y_\emptyset &= 1.
\end{aligned}
$$

Let $t \geq 1$. The *level-t Sherali–Adams relaxation for graph isomorphism* has variables y_I ranging over $\mathbb{R}$ for $I \in \binom{V(G)\times V(H)}{\leq t}$. The constraints are

$$\sum_{h \in V(H)} y_{I \cup \{gh\}} = y_I \text{ for all } I \text{ s.t. } |I| \leq t-1 \text{ and all } g \in V(G),$$
$$\sum_{g \in V(G)} y_{I \cup \{gh\}} = y_I \text{ for all } I \text{ s.t. } |I| \leq t-1 \text{ and all } h \in V(H),$$
$$y_I = 0 \text{ if } I \text{ s.t. } |I| \leq t \text{ is not partial isomorphism}$$
$$y_\emptyset = 1.$$

Extra: Graph Classes

$$
\begin{tikzcd}
\mathcal{OP} \arrow[dr] & \mathcal{PW}_{2t-1} \arrow[d] & \mathcal{TW}_{\max\{2t-1,2\}} \arrow[d] & \\
\mathcal{TW}_{t-1} \arrow[r] & \mathcal{L}_t \arrow[r] & \mathcal{L}_t^+ \arrow[r] & \mathcal{TW}_{3t-1}
\end{tikzcd}
$$

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