

FLIP-WIDTH

COPS AND ROBBER ON DENSE GRAPHS

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bounded treewidth



excluding a minor



excl. a topological minor



bounded expansion



nowhere dense



Theorem (Courcelle 1990)

Model checking Monadic Second Order logic is fpt
on every class of bounded treewidth
exhibit good
algorithmic,
combinatorial,
& logical behavior

Theorem (Grohe, Kreutzer, Siebertz, 2017)

Model checking First-Order logic is fpt
on every nowhere dense class.

Furthermore, for a monotone graph class C ,
model checking FO is fpt $\Leftrightarrow C$ is nowhere dense

BEYOND SPARSE

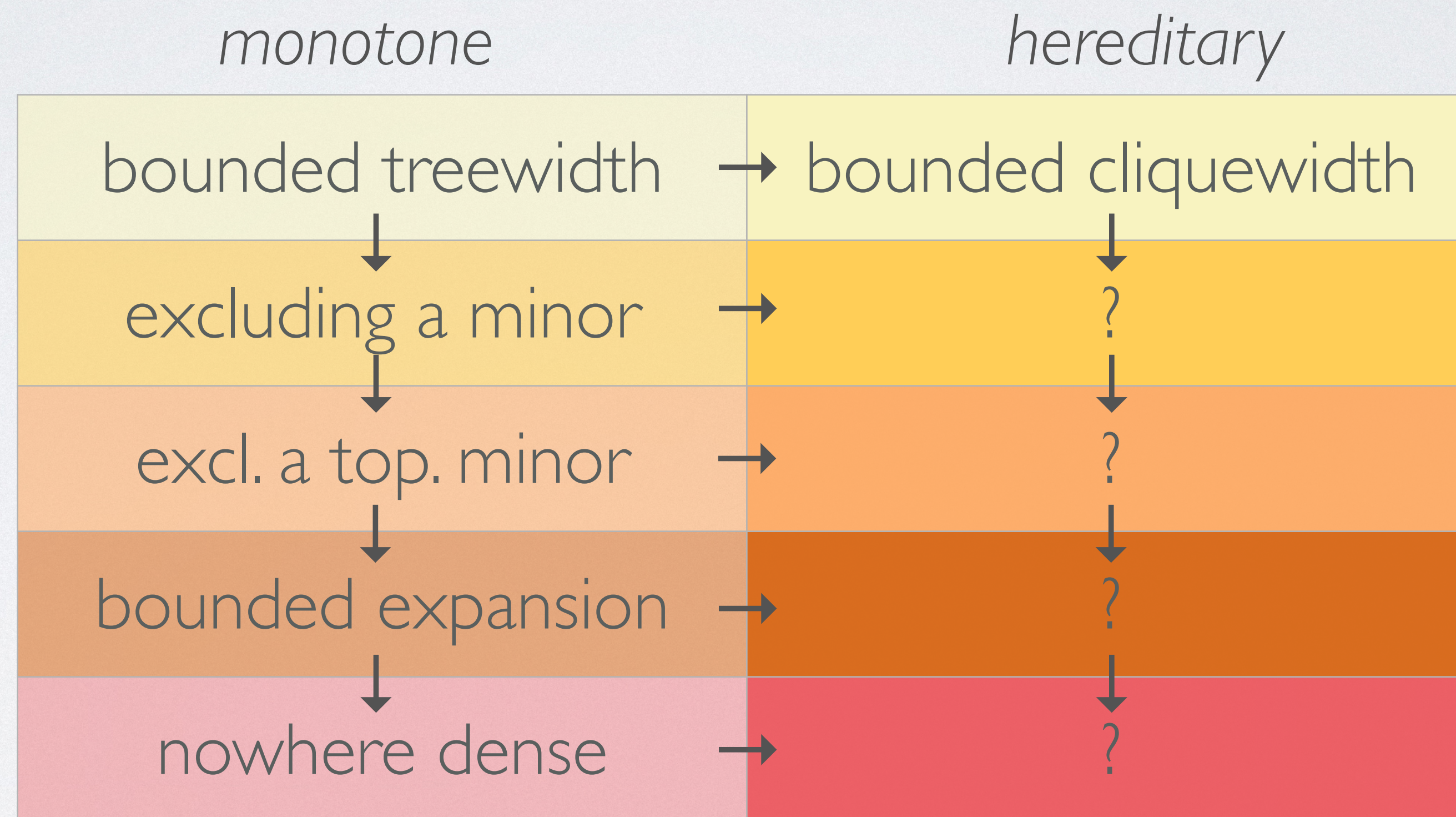
Example: treewidth $\rightarrow$ cliquewidth/rankwidth

- retain many good properties of treewidth
- applicable to dense graphs

Theorem (Courcelle, Rotics, Makowsky 2000+Oum, Seymour 2006)
Model checking MSO is fpt on classes of bounded rankwidth

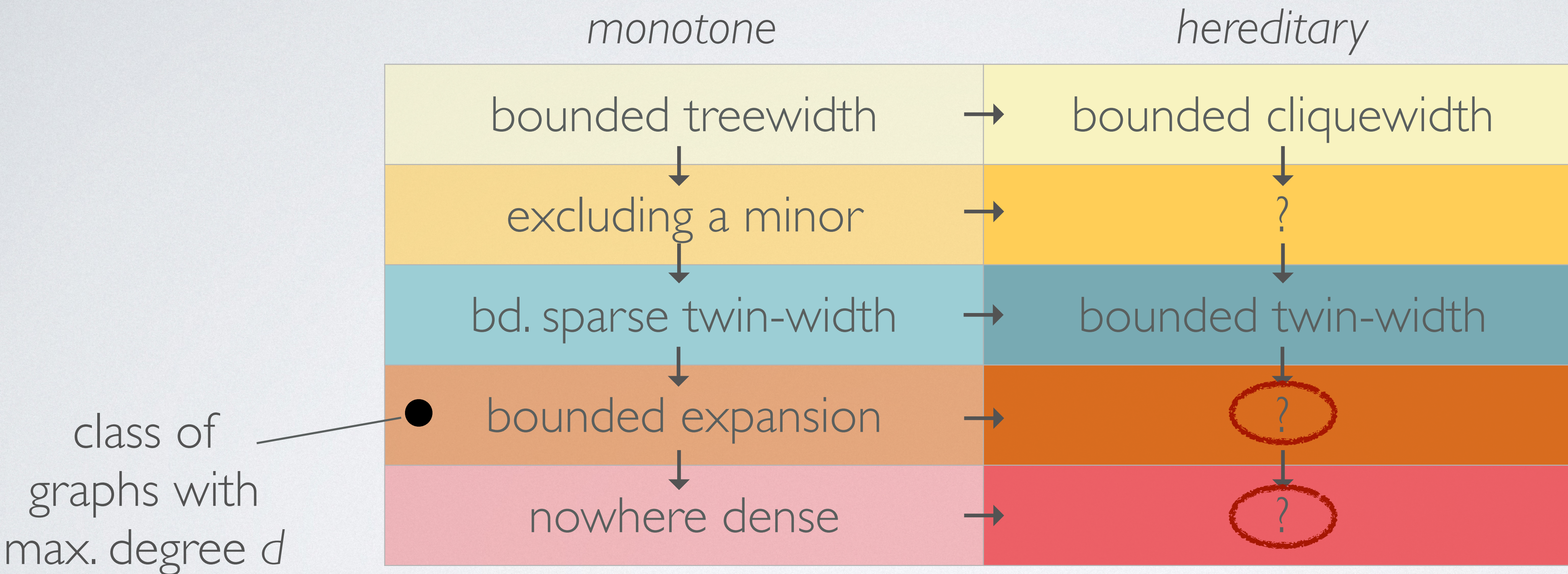
BEYOND SPARSITY

Project: extend from monotone classes to hereditary classes



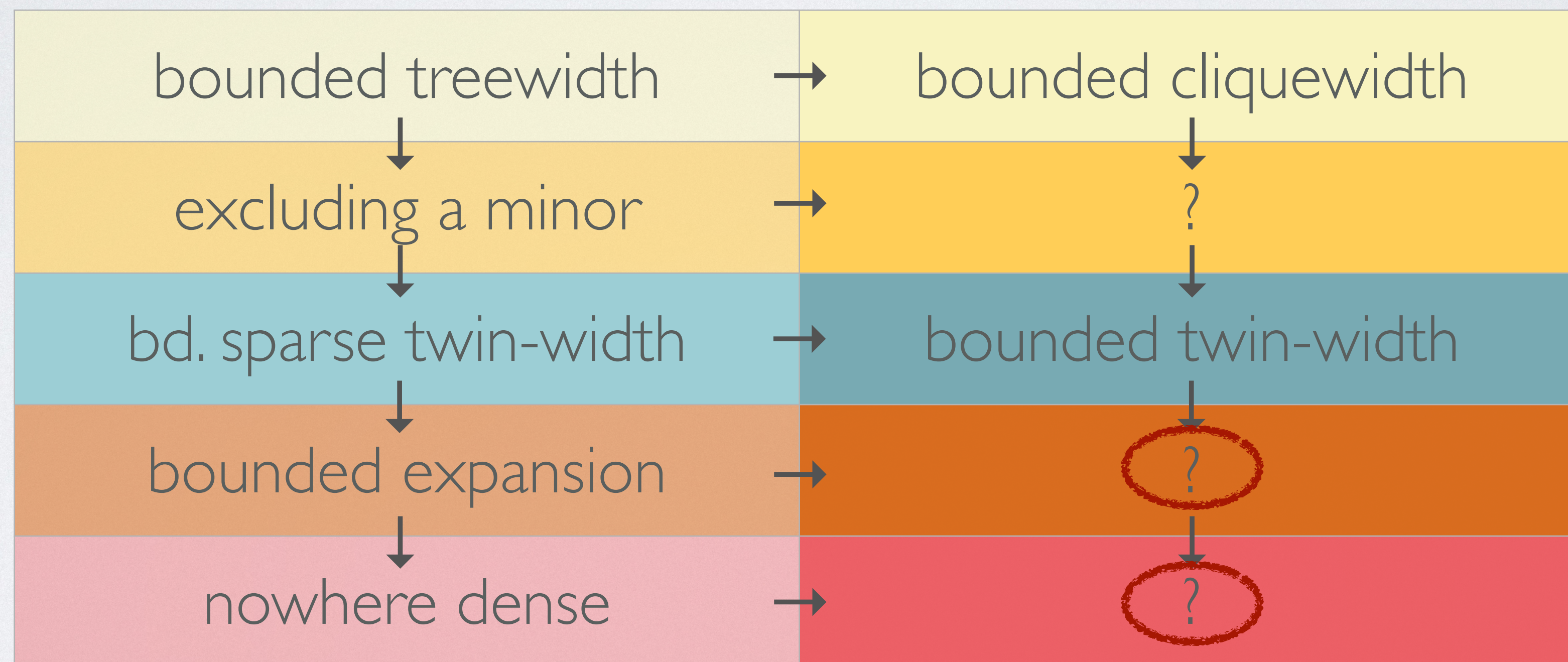
Quest (Grohe): which hereditary classes have fpt model checking?

TWIN-WIDTH



GRAND UNIFICATION

Common generalization of Sparsity theory and Twin-width



WANTED

Analogues of the fundamental parameters
studied in Sparsity Theory:

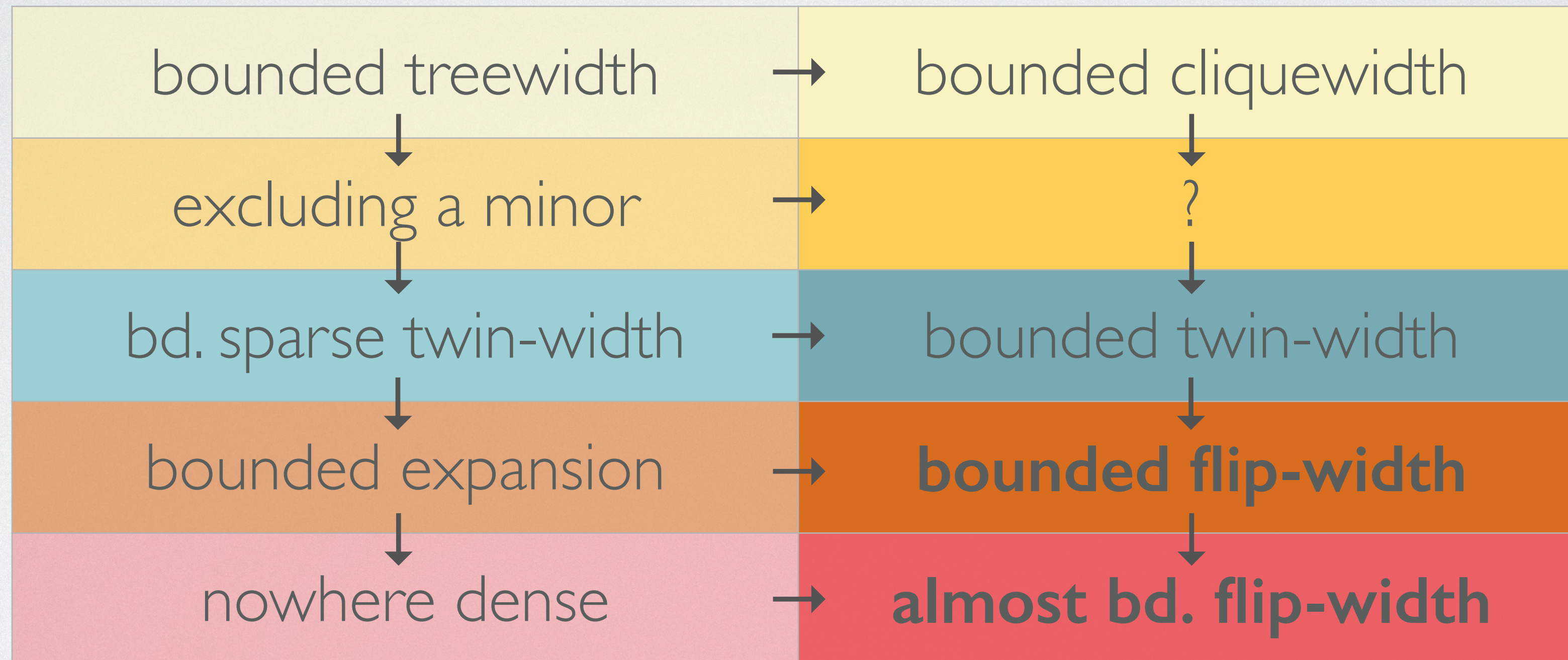
degeneracy

generalized coloring numbers:
star-chromatic number, weak coloring numbers

CONTRIBUTION

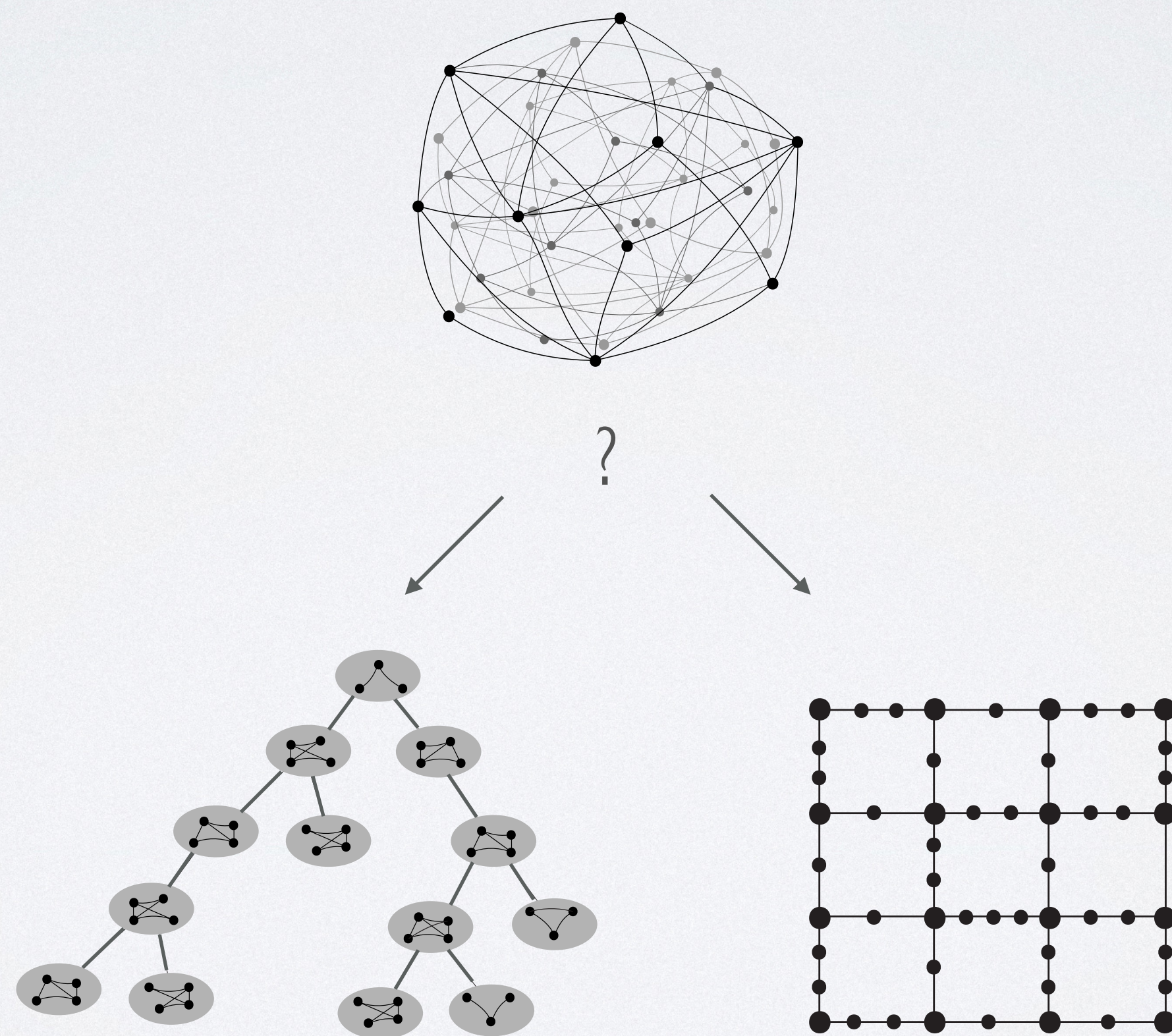
- *flip-width* parameters
 - dense analogues of the fundamental parameters studied in sparsity theory
 - include degeneracy, twin-width, and clique-width as a *special cases*.
- notion of classes of *bounded flip-width* and *almost bounded flip-width*

GRAND UNIFICATION?



HOW TO DEFINE GRAPH PARAMETERS

- through decompositions
- through obstructions

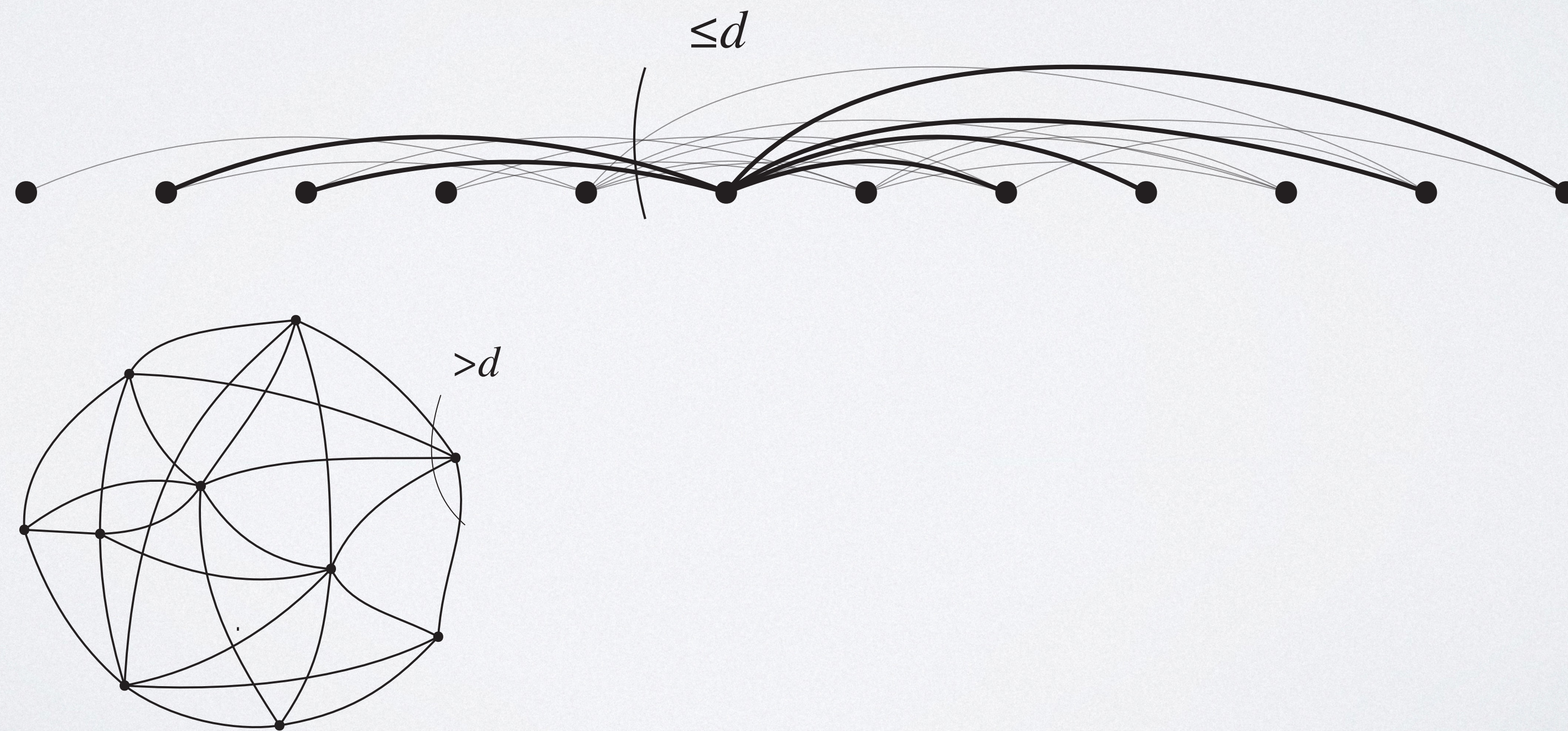


DEGENERACY

A graph is d -degenerate
if every subgraph has a vertex of degree $\leq d$

Or: there is an ordering such that every vertex has $\leq d$ neighbors before it

dichotomy:



HOW TO DEFINE GRAPH PARAMETERS

- through decompositions
- through obstructions
- through games

COPS AND ROBBER

Seymour and Thomas (1993)

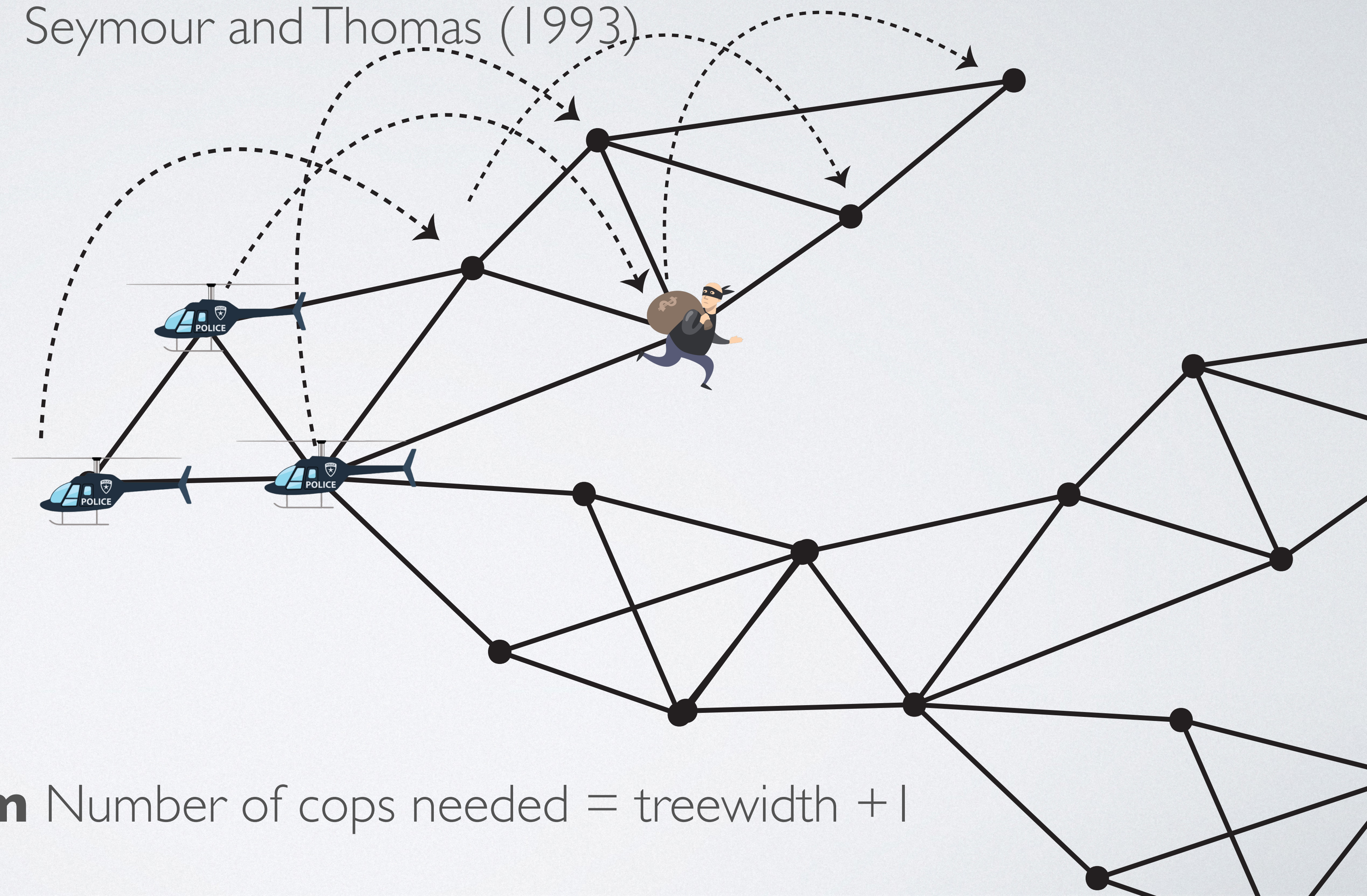
k cops



a robber



moves along edges
at *infinite speed*



Theorem Number of cops needed = treewidth + 1

SPEED LIMIT



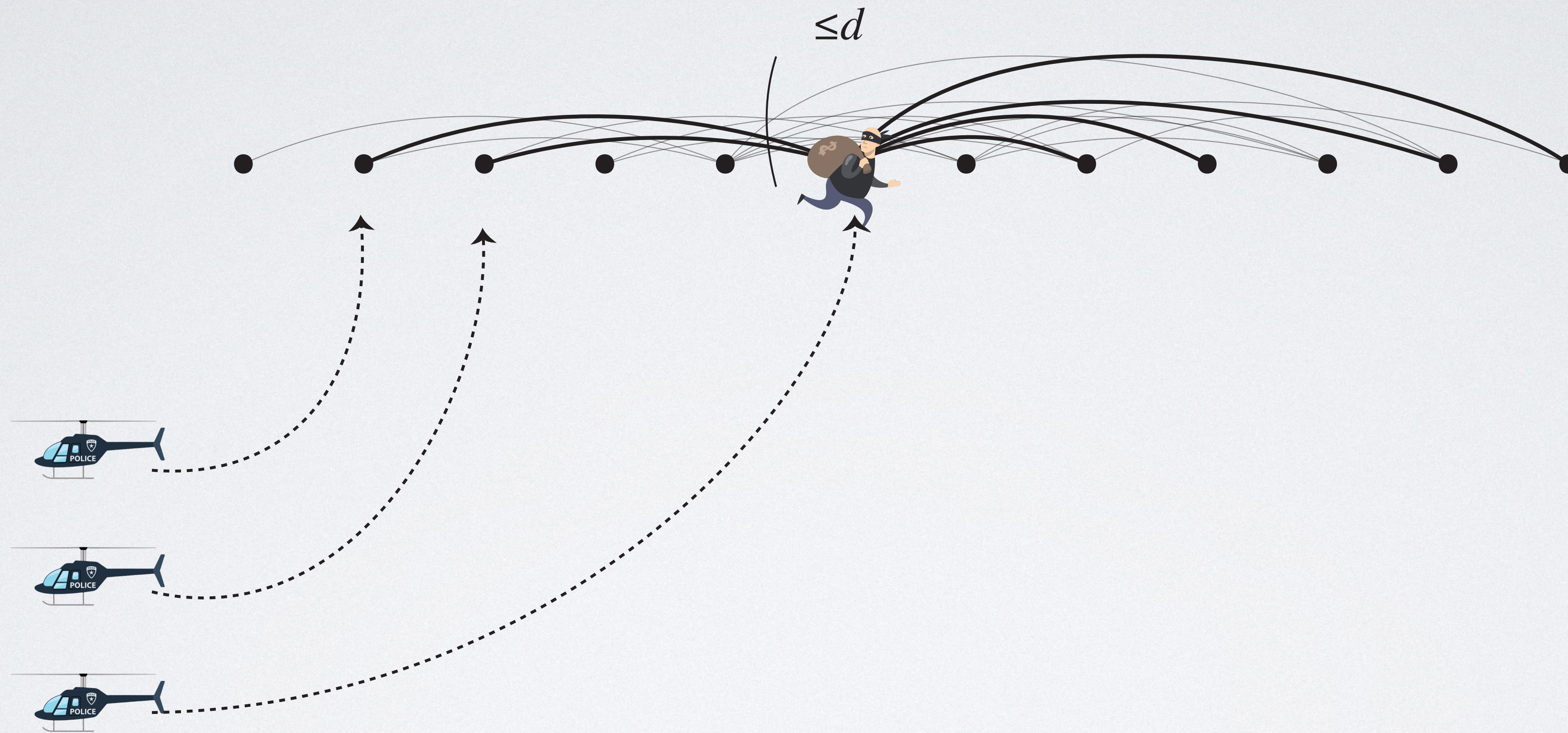
robber moves at speed r

$\text{copwidth}_r(G) :=$ number of cops needed to capture robber

Fact. $\text{copwidth}_1(G) = \text{degeneracy}(G) + 1$

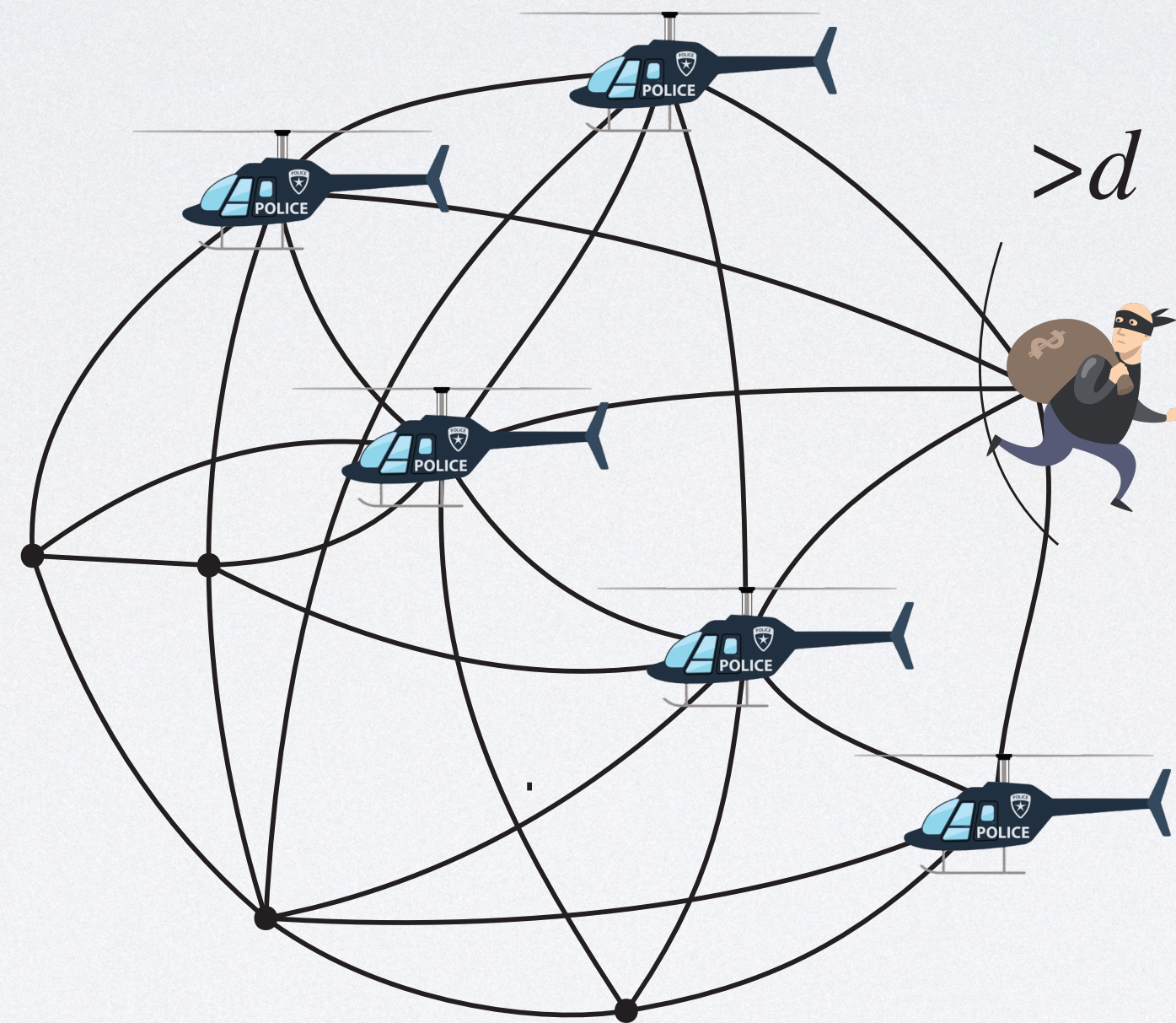
[$\approx$ Richerby & Thilikos 2008]

$$\text{copwidth}_1(G) \leq \text{degeneracy}(G) + 1$$



$$\text{copwidth}_1(G) \geq \text{degeneracy}(G) + 1$$

G is not d -degenerate $\rightarrow d+1$ cops do not suffice



Theorem

$$\text{copwidth}_1(G) = \text{degeneracy}(G) + 1$$

$$\text{copwidth}_\infty(G) = \text{treewidth}(G) + 1$$

$$\text{copwidth}_r(G) \approx \Theta(r)\text{-weak coloring number}(G)$$

Theorem. Let $\mathcal{C}$ be a graph class. Then:

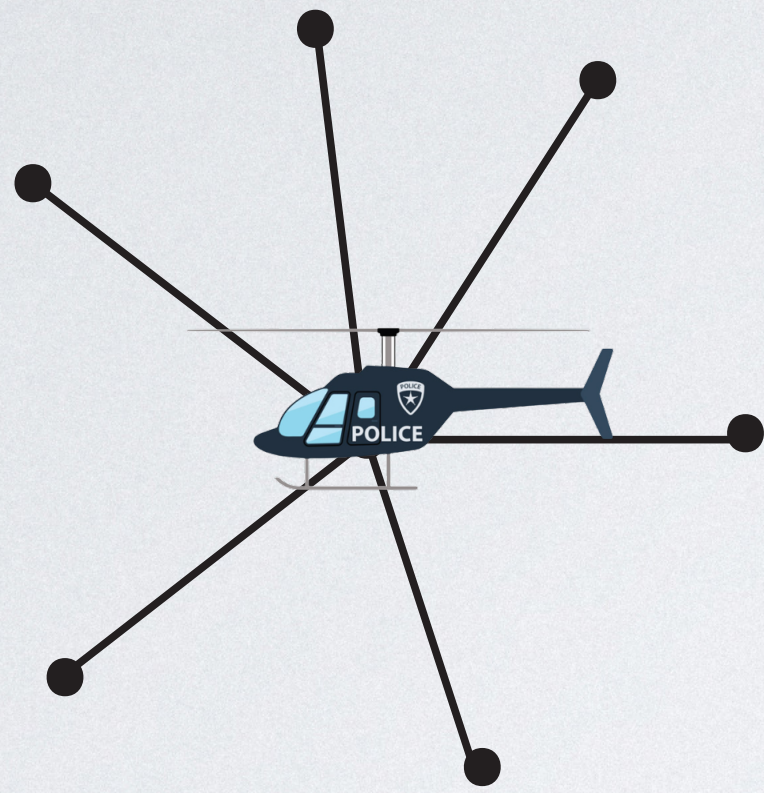
$\mathcal{C}$ has bounded expansion



for all $r \in \mathbf{N}$, $\text{copwidth}_r(\mathcal{C}) < \infty$

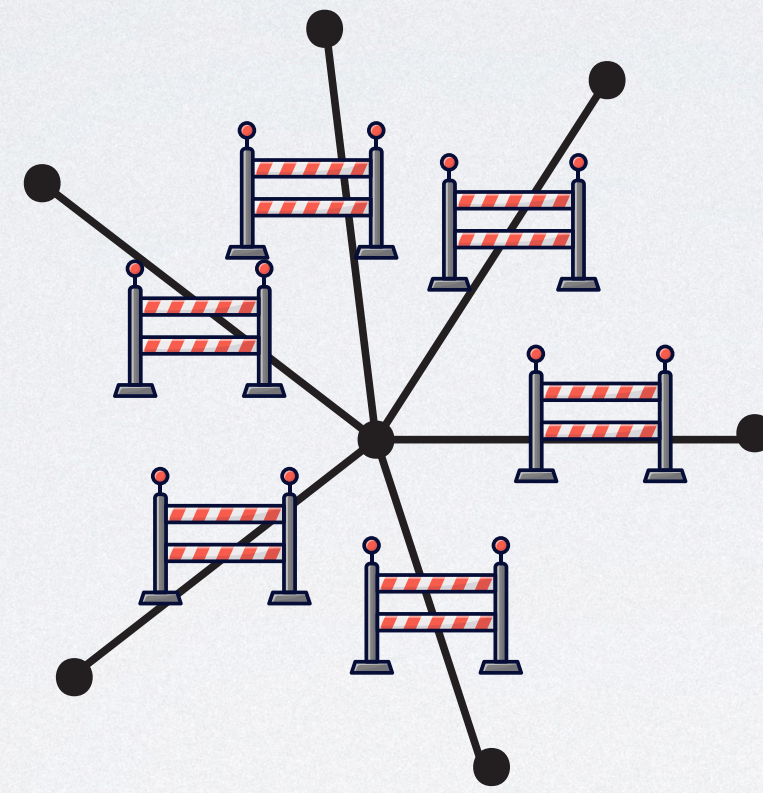
$$\sup_{G \in \mathcal{C}} \text{copwidth}_r(G)$$

GOING DENSE



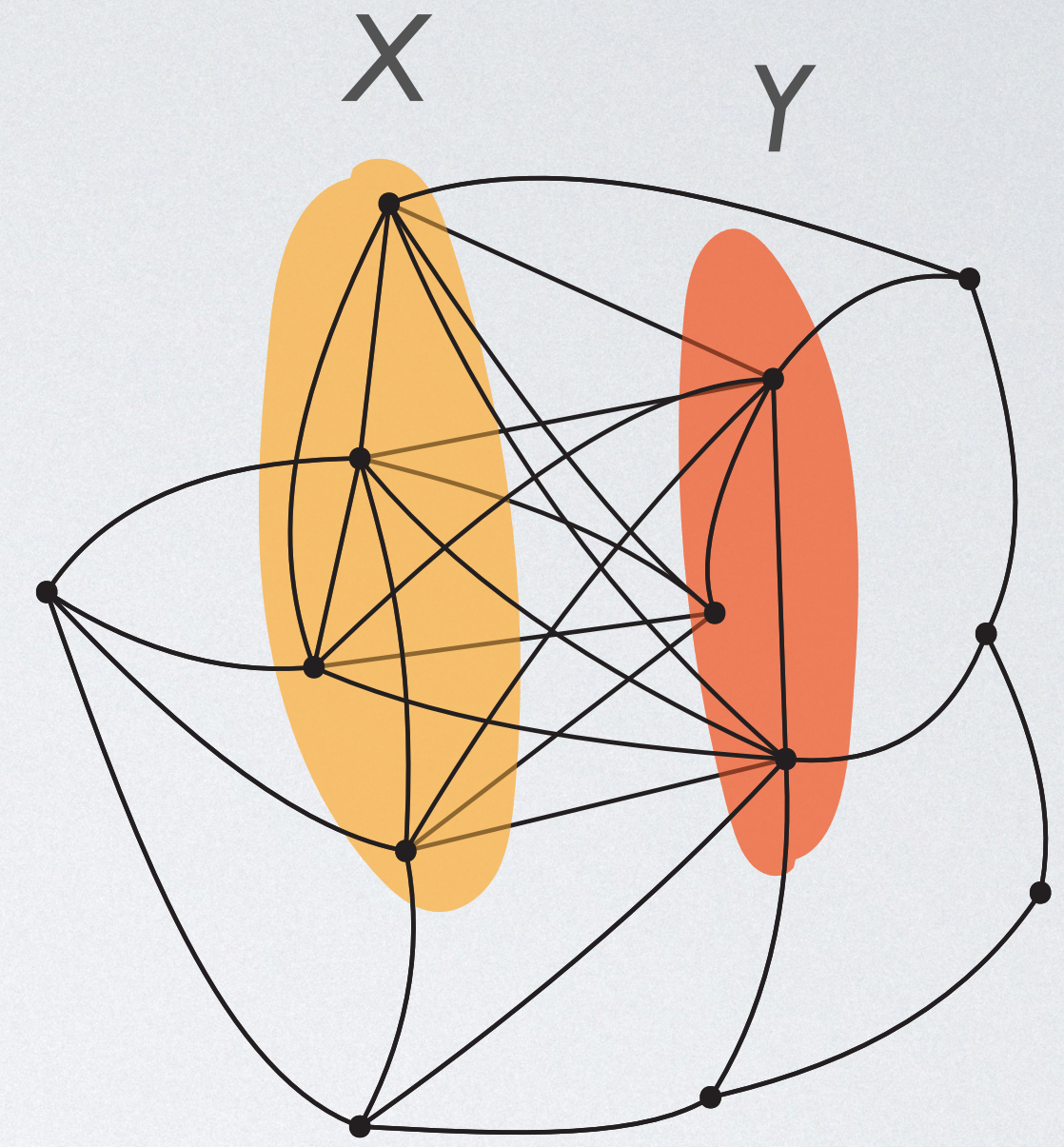
blocking a vertex

=



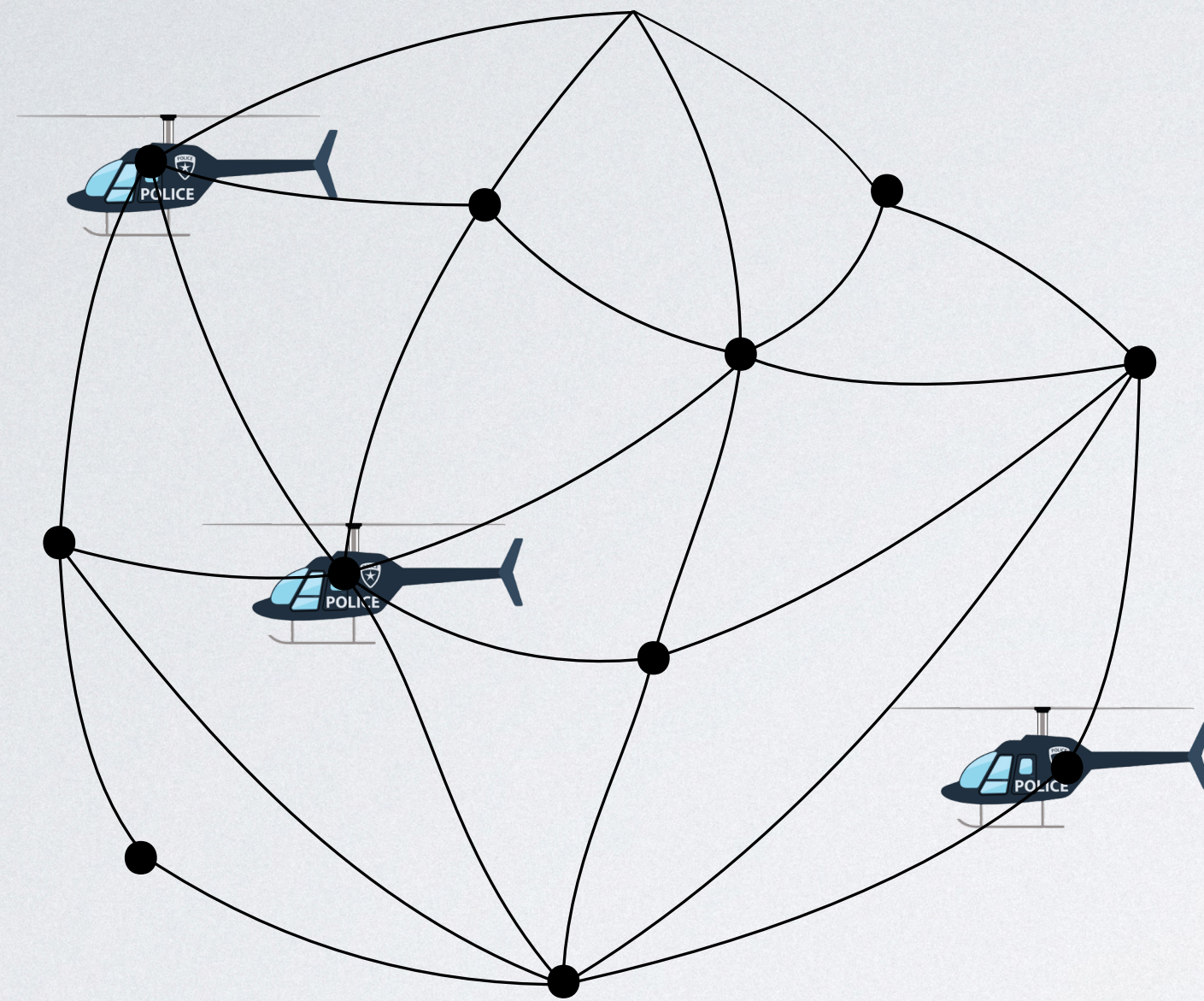
isolating a vertex

→

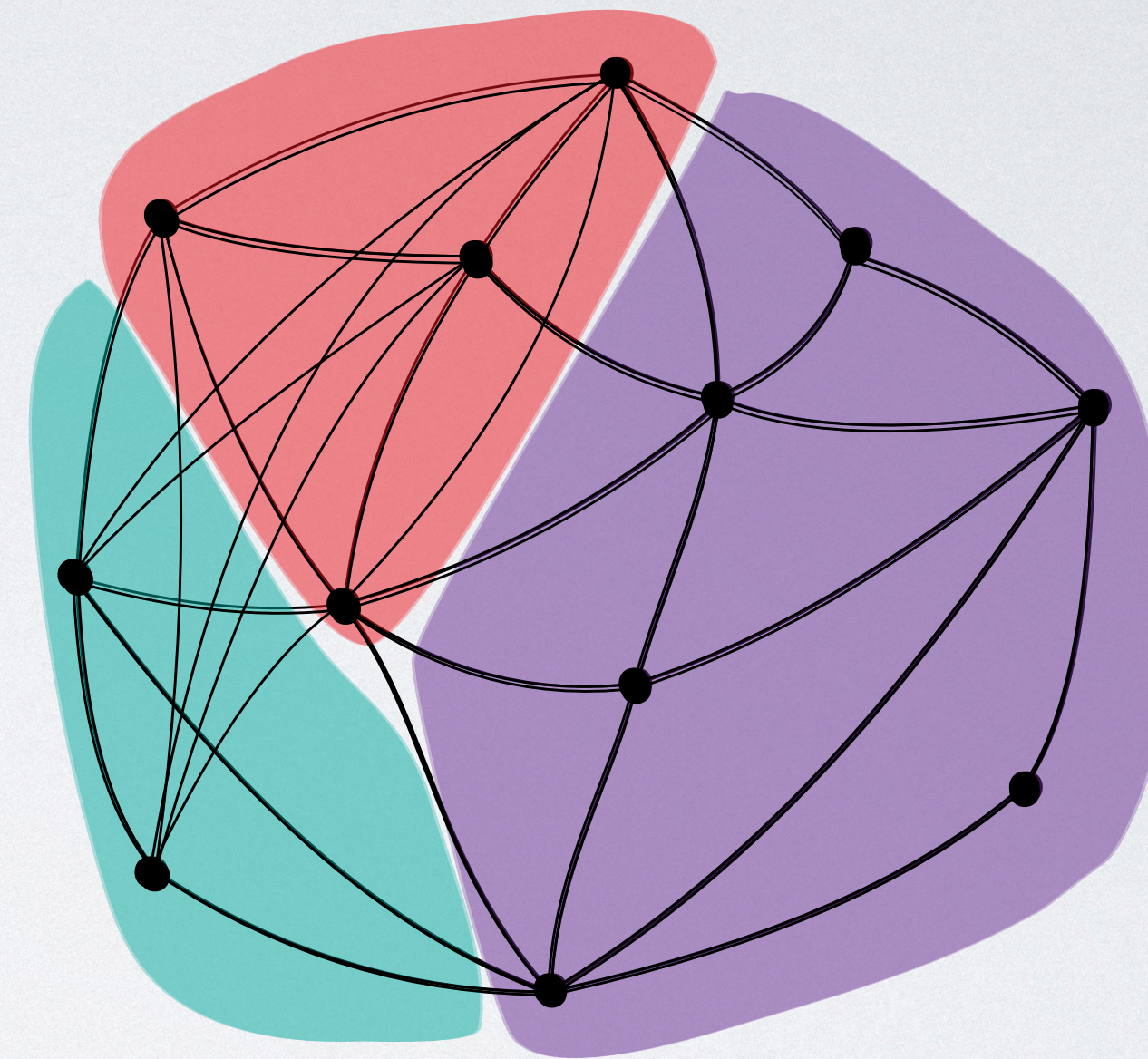
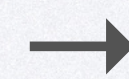


flip X and Y in G

GOING DENSE



blocking k vertices



k-flip of G :

partition $V(G) = A_1 \cup \dots \cup A_k$

For each pair $A_i A_j$

flip or not

FLIPPER GAME

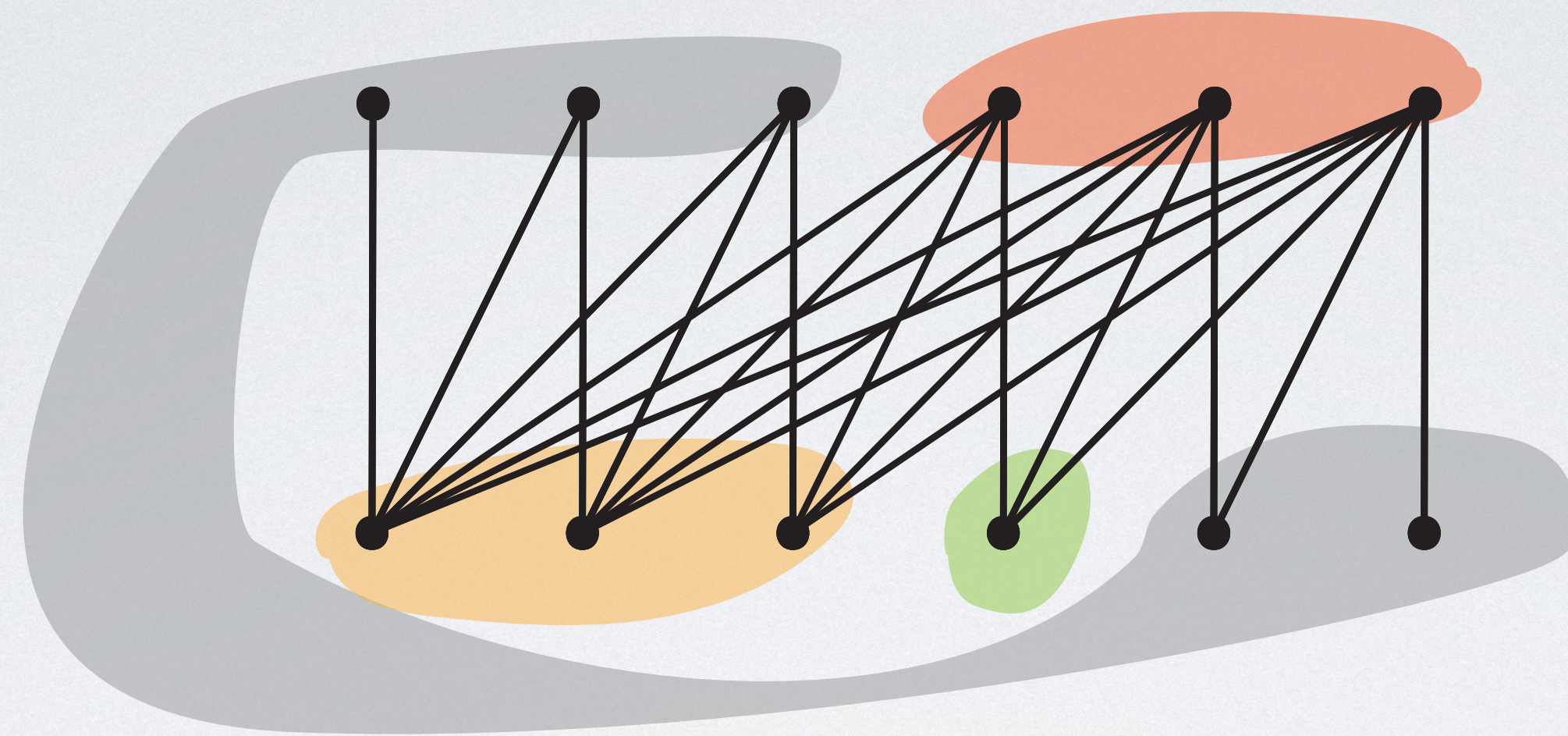
With radius r and flip power k

In each round:

- Flipper announces *next* k -flip G_k of G
- Then runner runs at speed r in *previous* k -flip G_{k-1} of G
- Runner loses if new position is isolated in G_k

$\text{flipwidth}_r(G) :=$ minimum k needed to capture runner

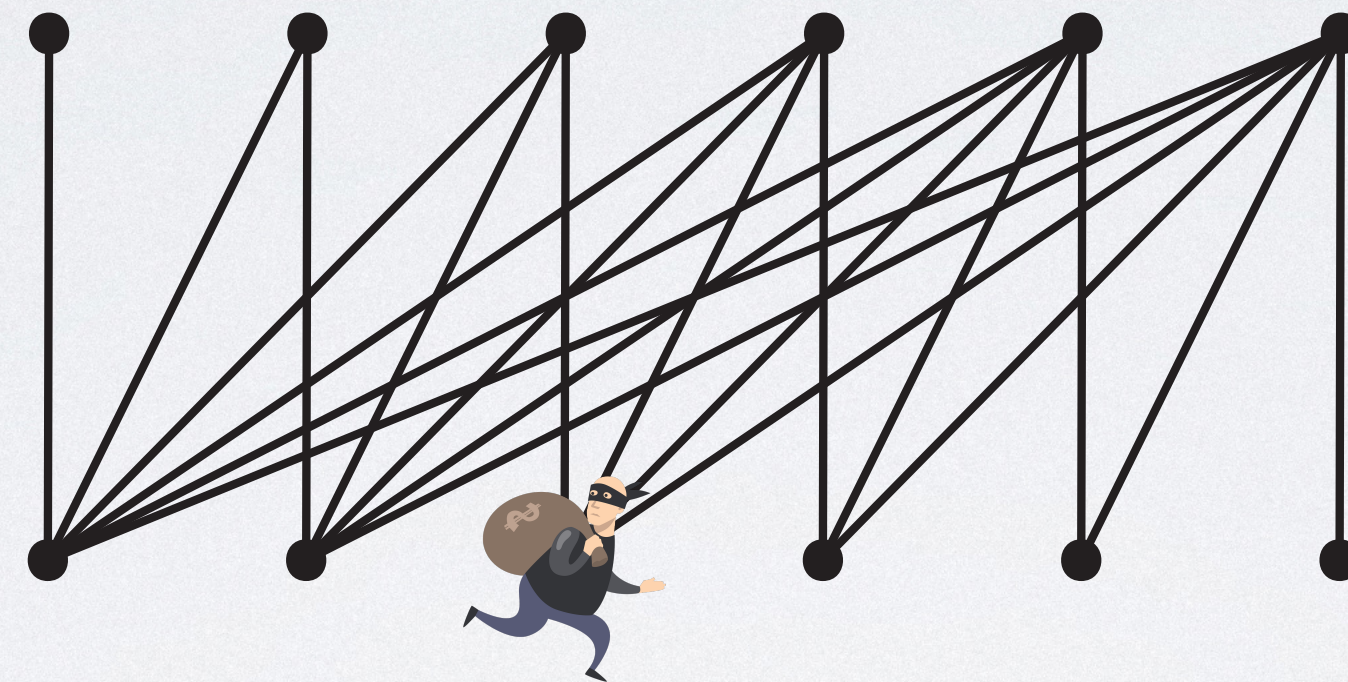
EXAMPLE



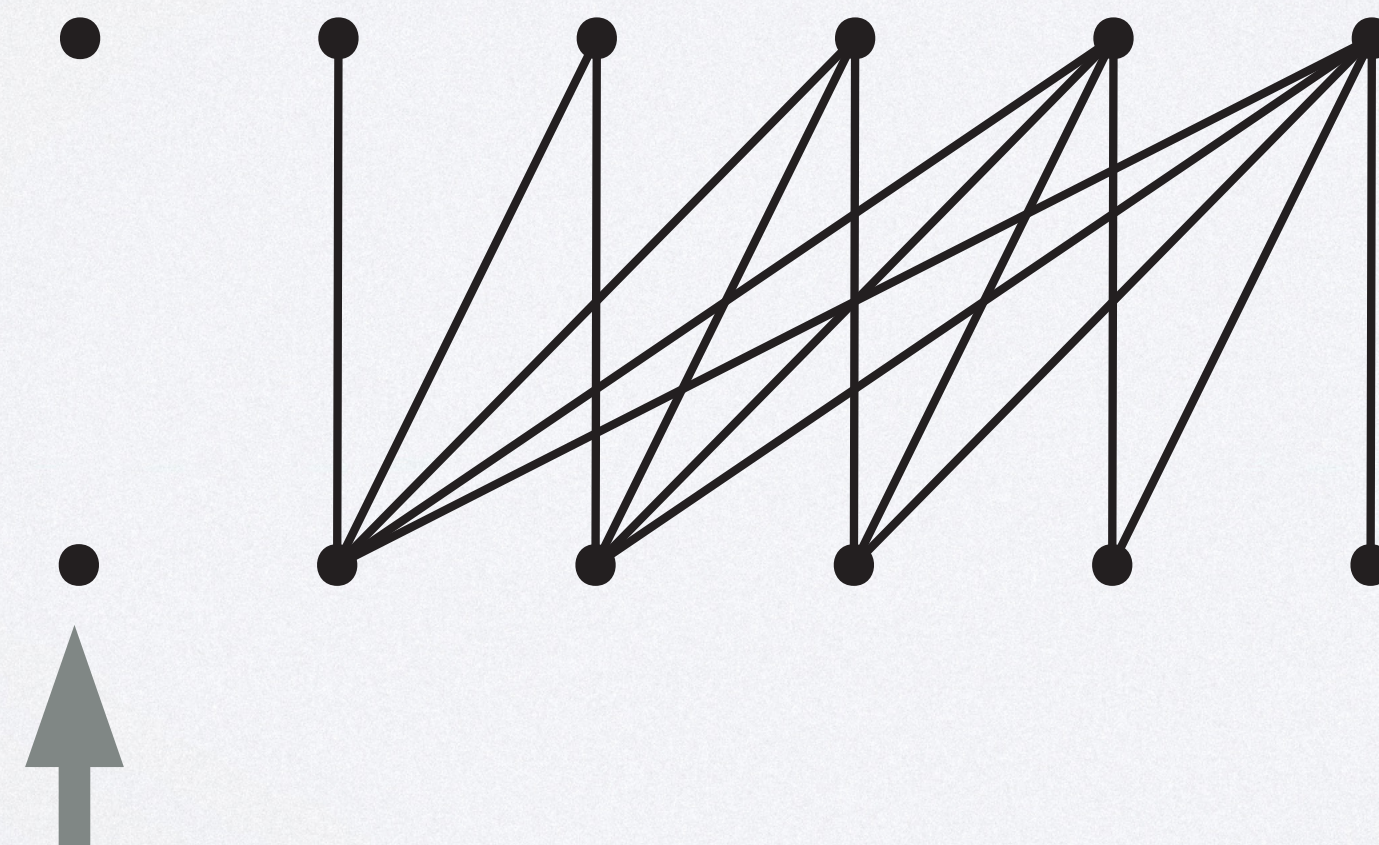
4-flip

EXAMPLE

speed $r = \infty$, flip power $k=4$

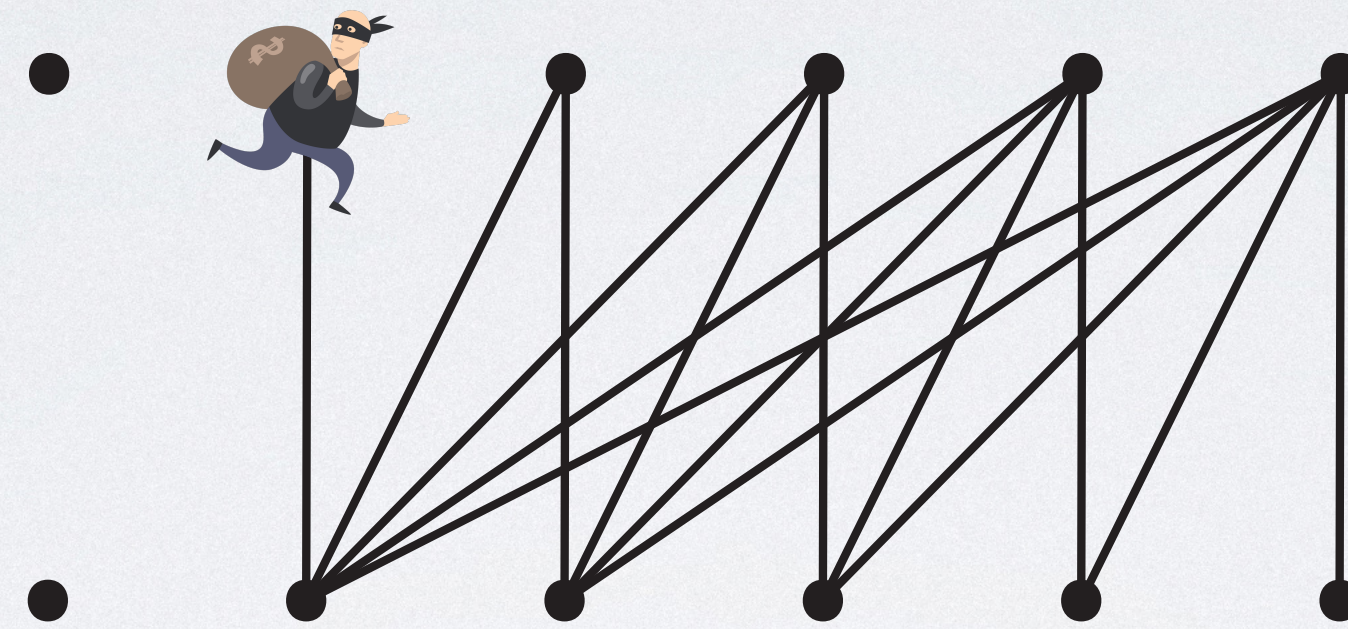


next flip:

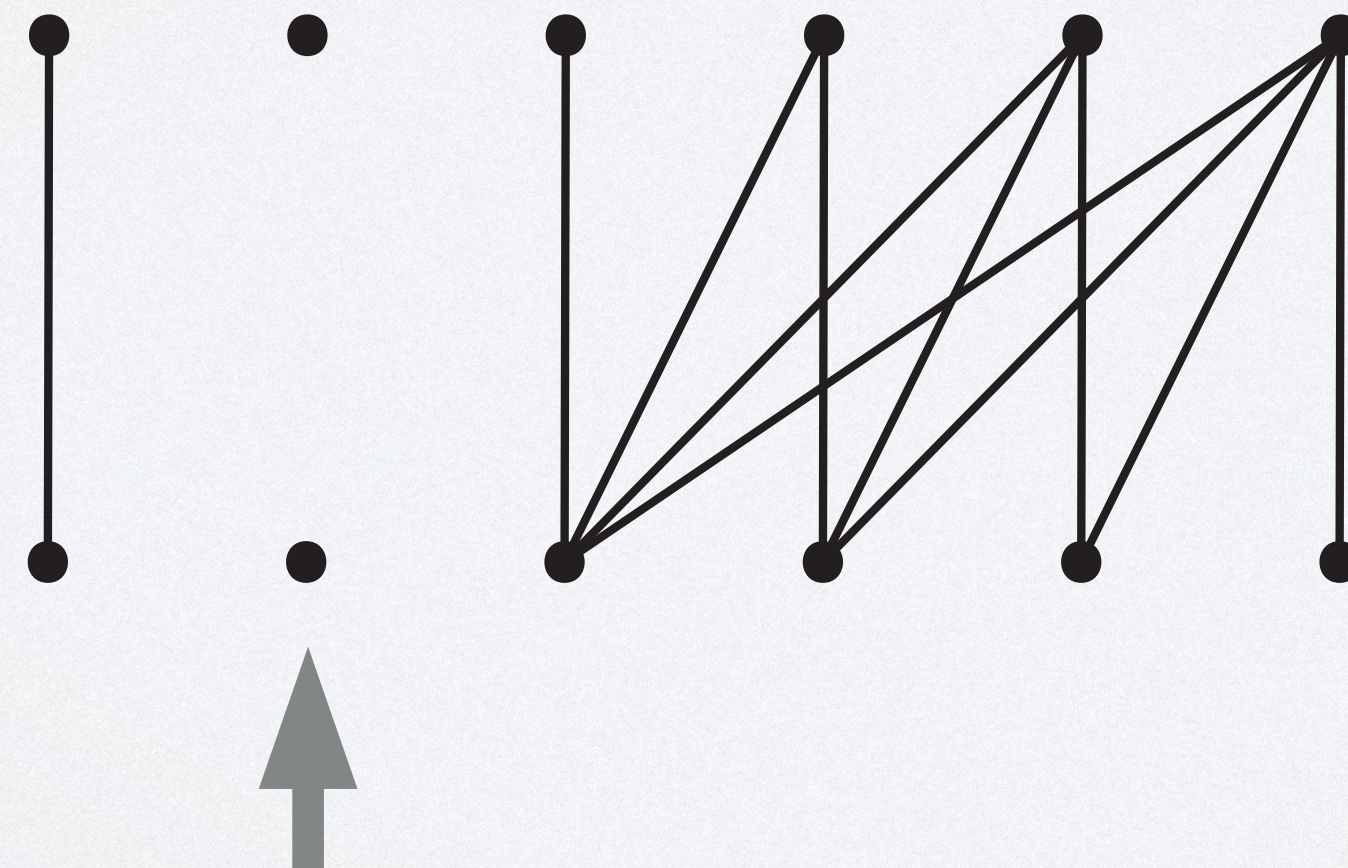


EXAMPLE

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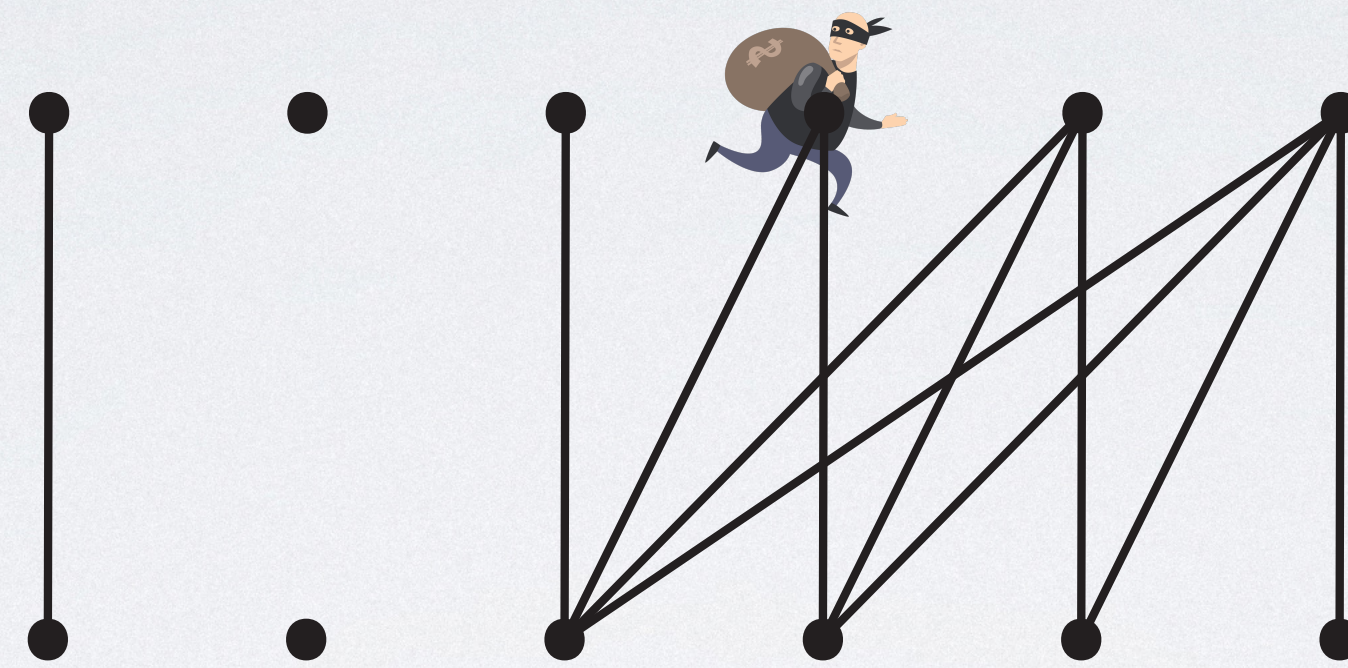


next flip:

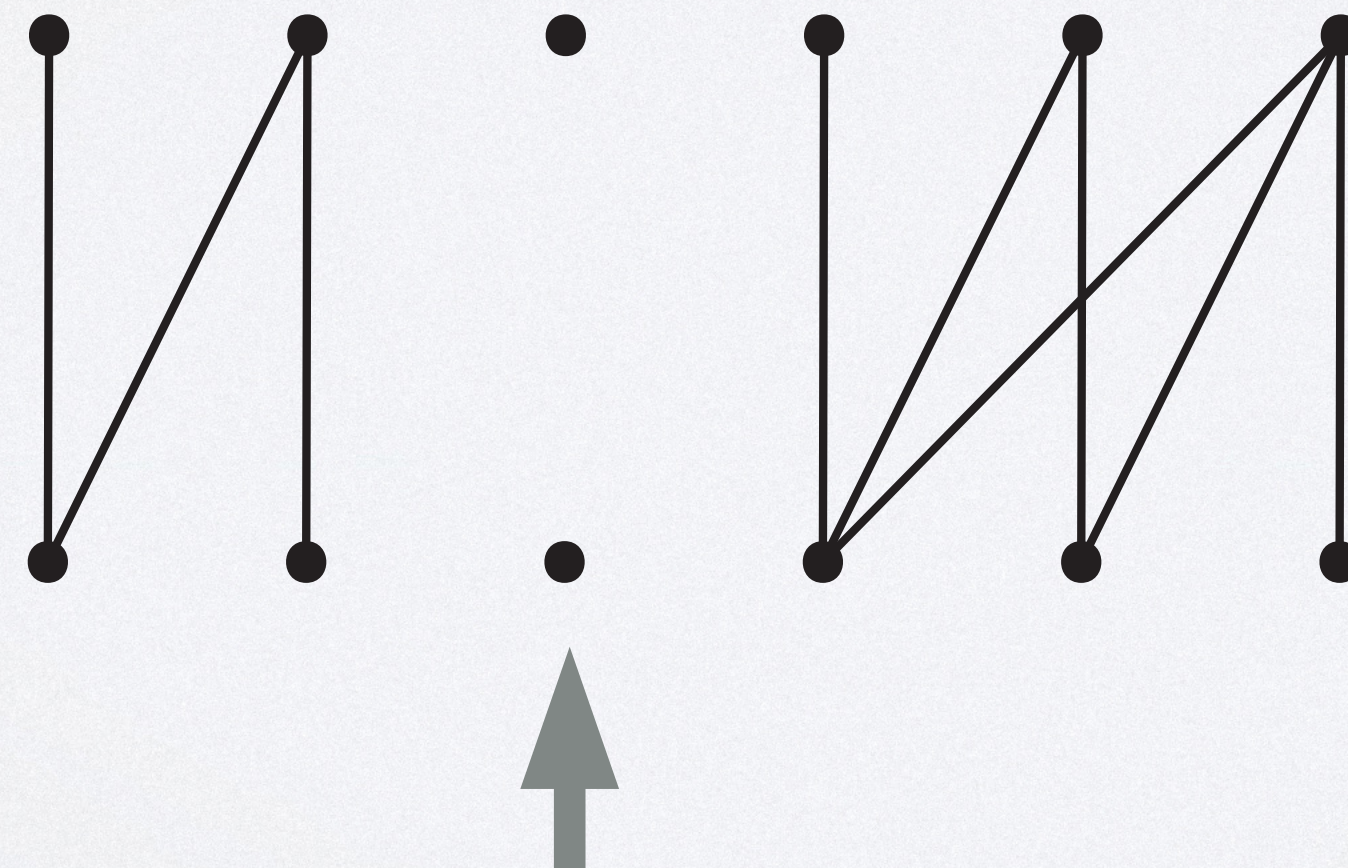


EXAMPLE

speed $r = \infty$, flip power $k=4$

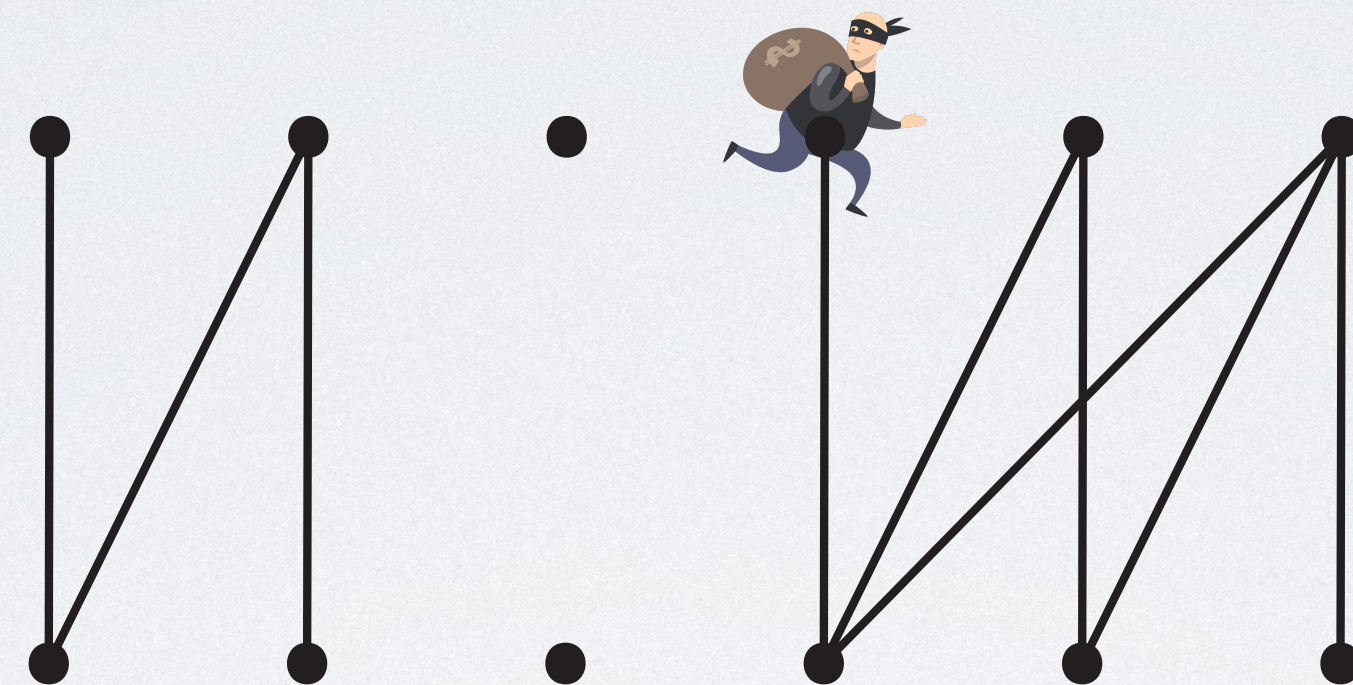


next flip:

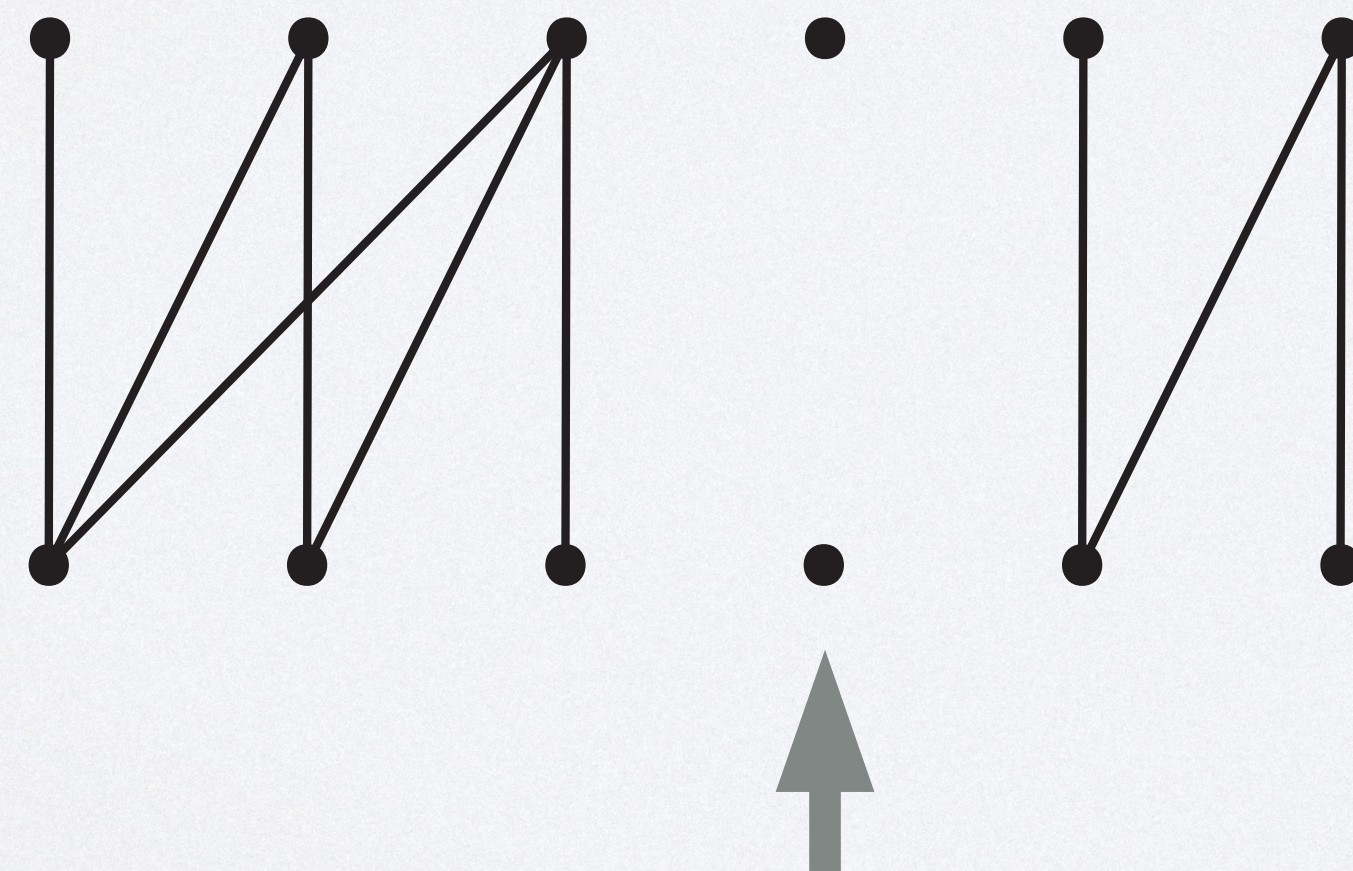


EXAMPLE

speed $r = \infty$, flip power $k=4$

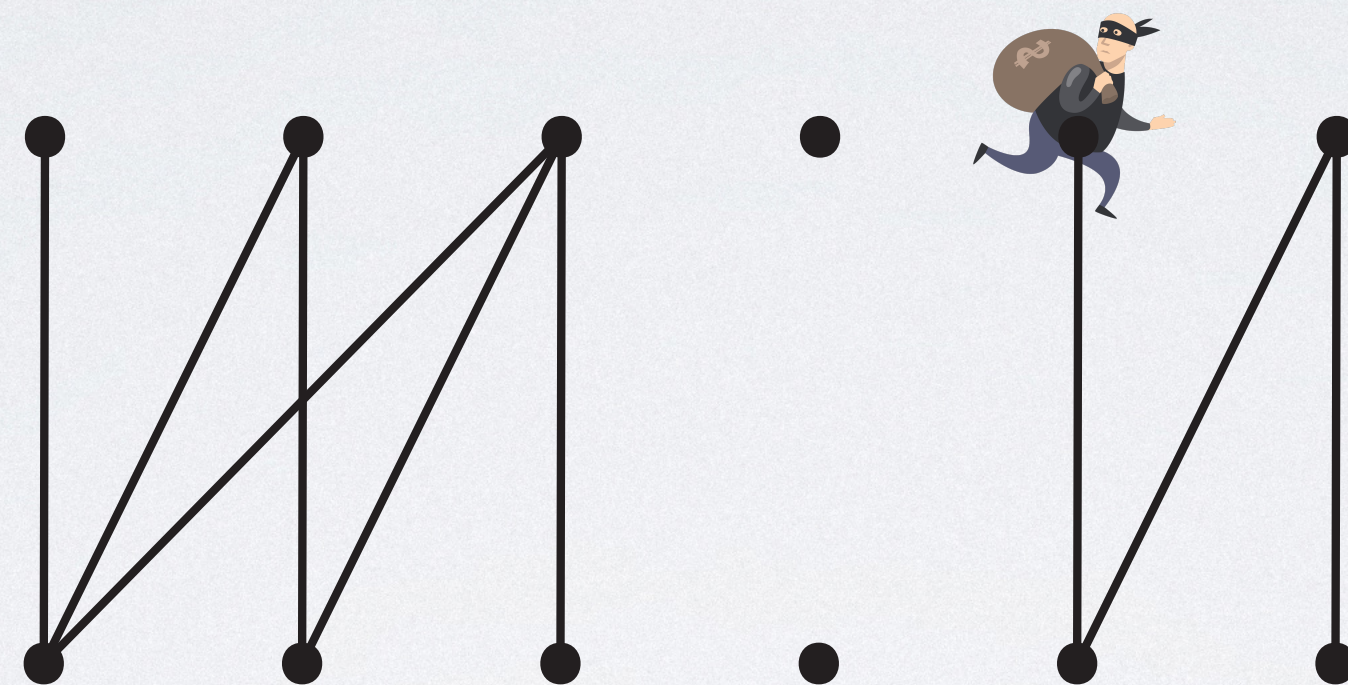


next 4-flip:

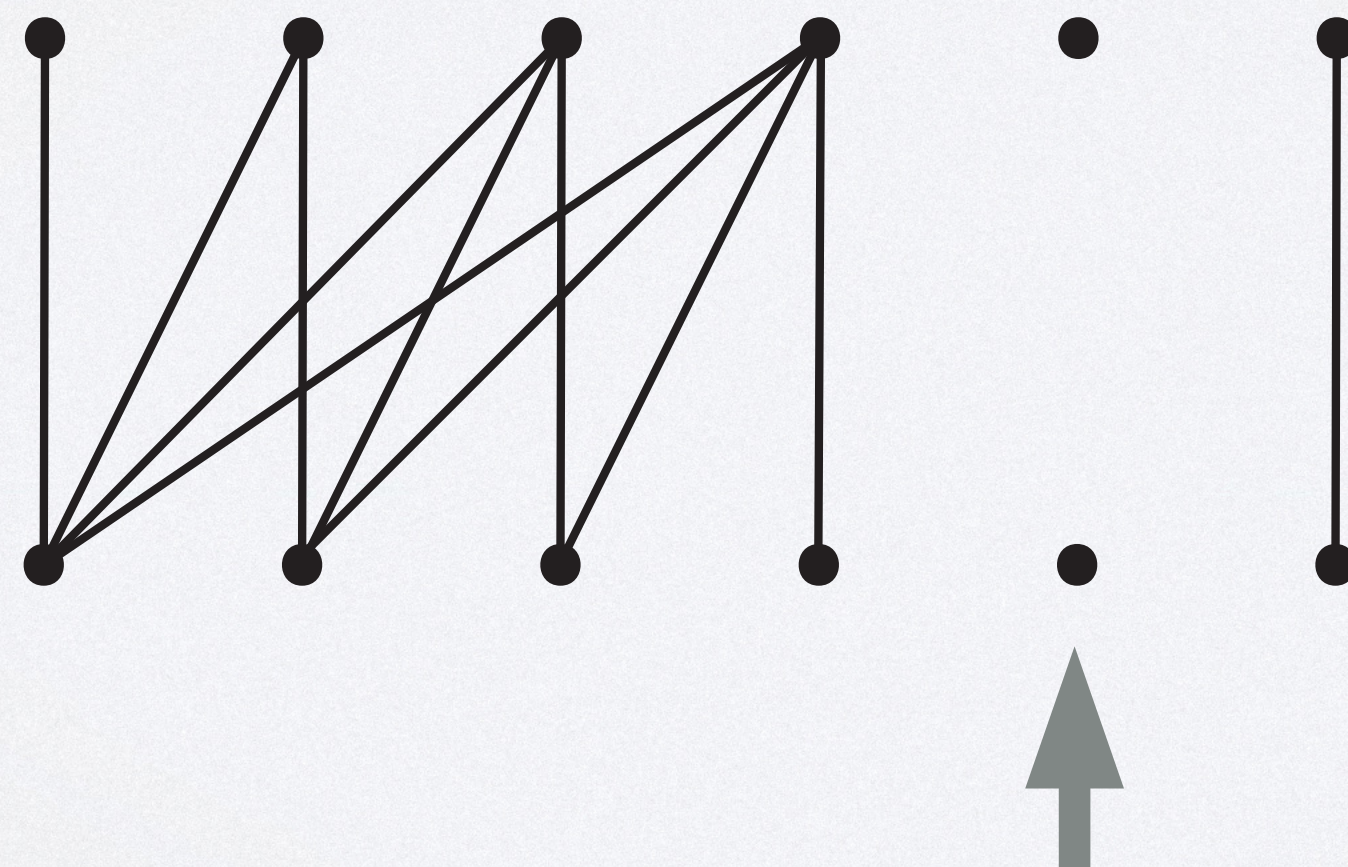


EXAMPLE

speed $r = \infty$, flip power $k=4$

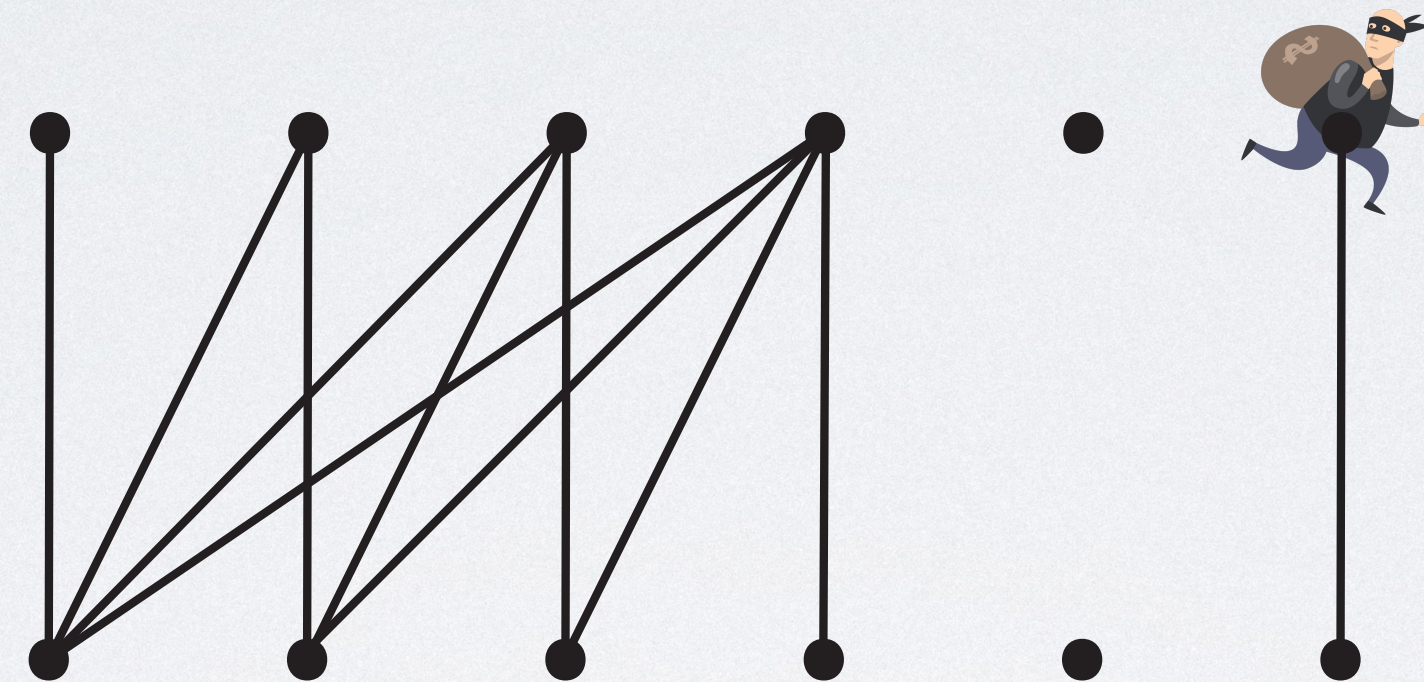


next 4-flip:

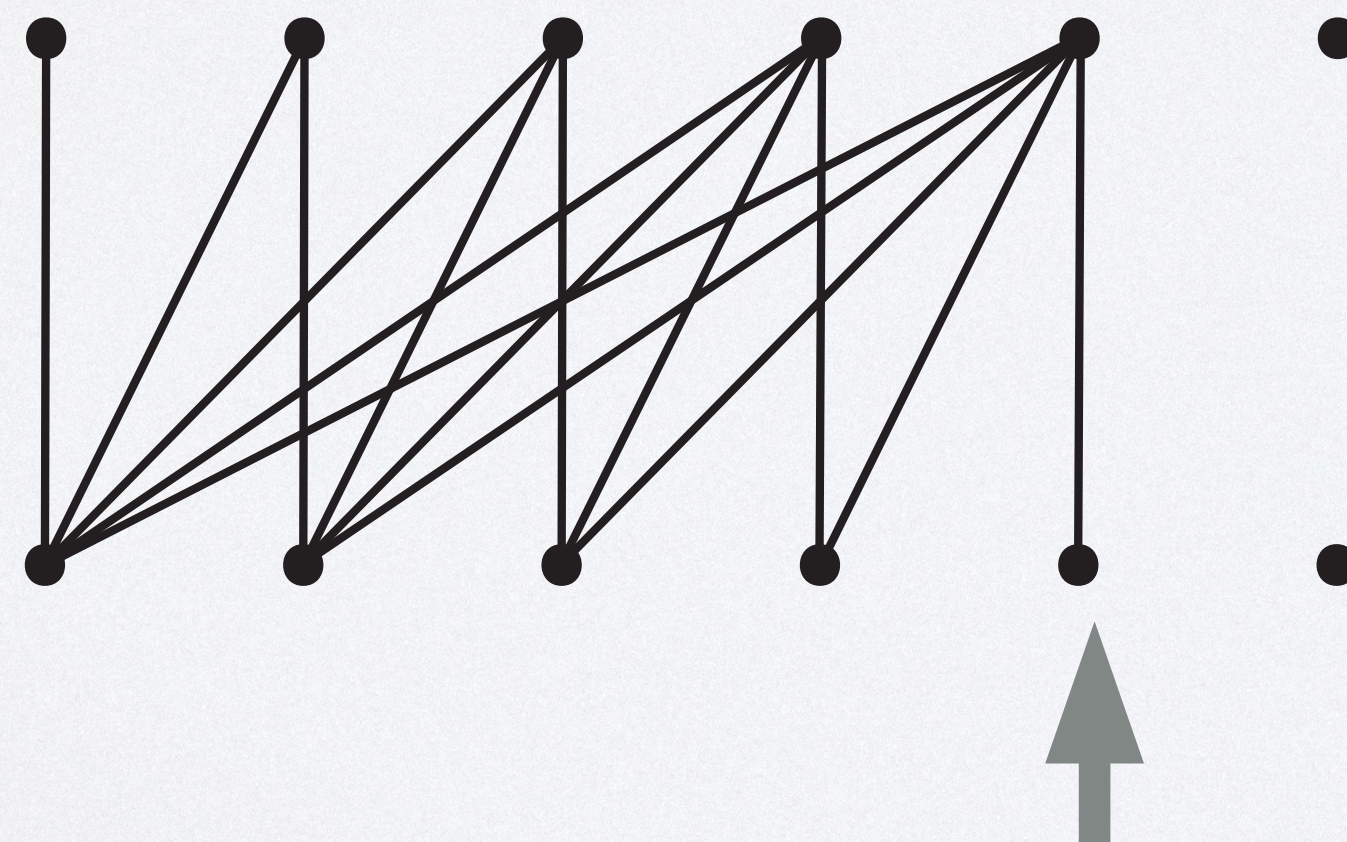


EXAMPLE

speed $r = \infty$, flip power $k=4$

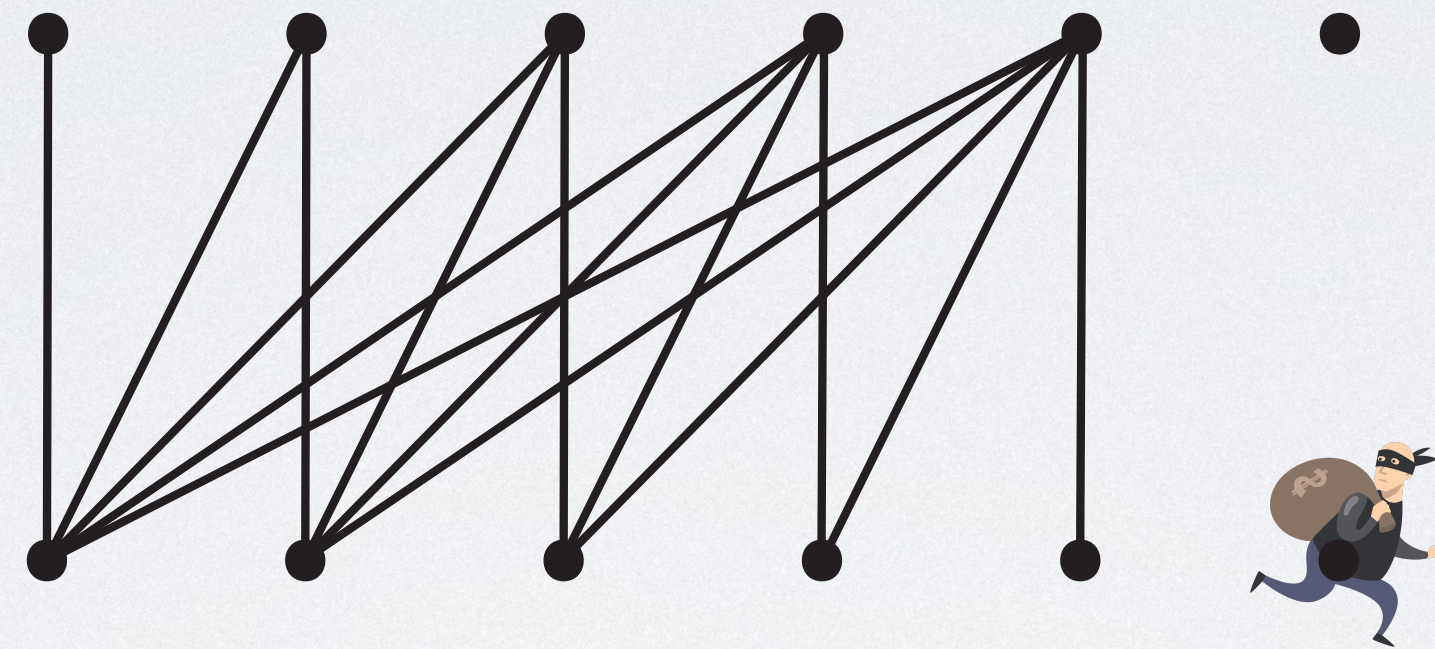


next 4-flip:



EXAMPLE

speed $r = \infty$, flip power $k=4$



$$\text{flip-width}(\text{graph}) \leq 4$$

RADIUS ∞

Theorem $\text{flip-width}_\infty(G) \approx \text{clique-width}(G)$

characterization of clique-width via games

Corollary A class $\mathcal{C}$ has bounded clique-width

$\Leftrightarrow$

$\text{flip-width}_\infty(\mathcal{C}) < \infty$

$$\text{rank-width}(G) \leq \text{flip-width}_\infty(G) \leq O(2^{\text{rank-width}(G)})$$

$$\text{rank-width}(G) \leq k$$



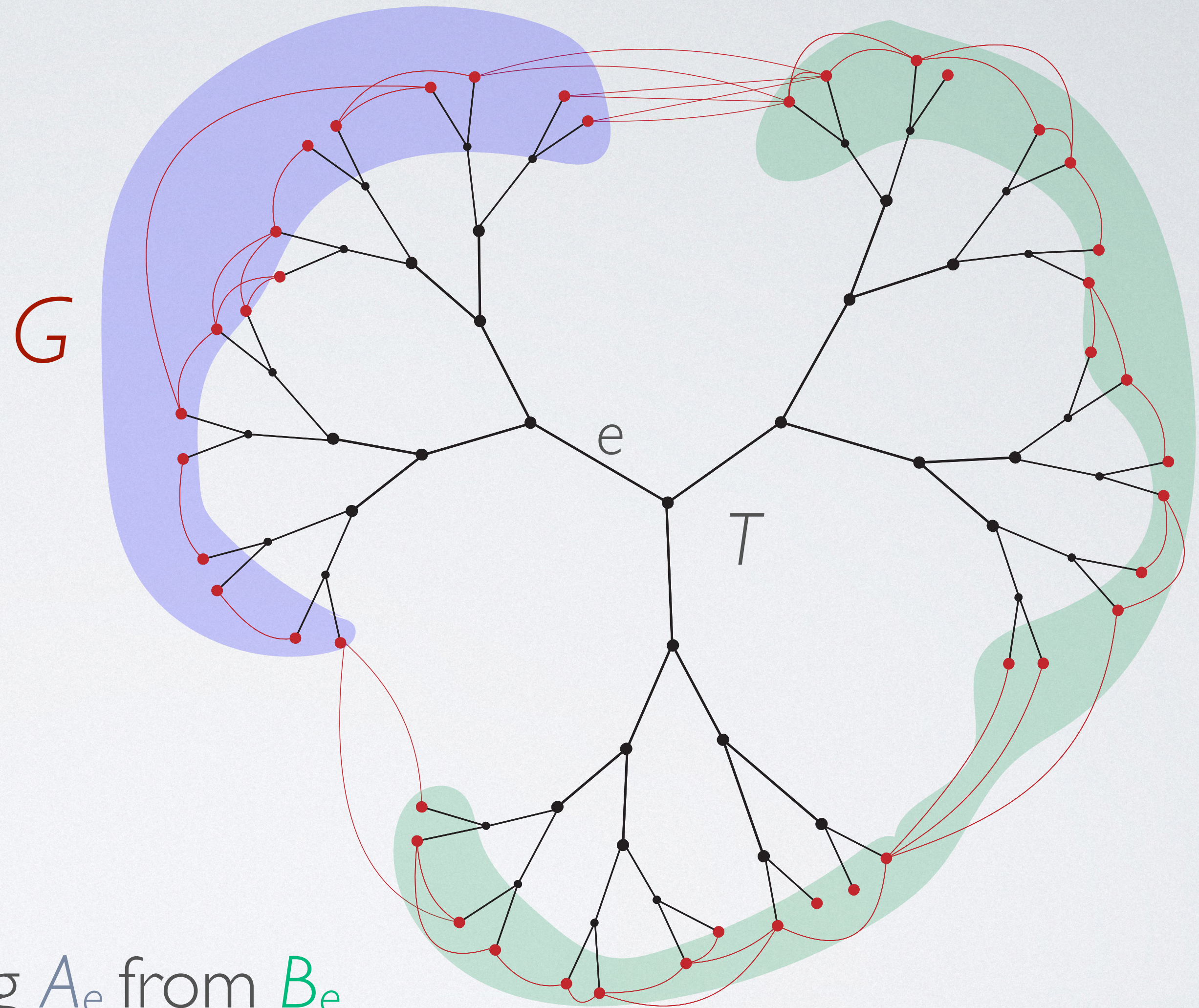
exists cubic tree T :

- $V(G) = \text{Leaves}(T)$
- For every edge e of T

$$V(G) = A_e \cup B_e$$

$$\text{Adj}_G[A_e, B_e] \text{ has rank } \leq k$$

$\Rightarrow \exists O(2^k)$ -flip G' of G separating A_e from B_e



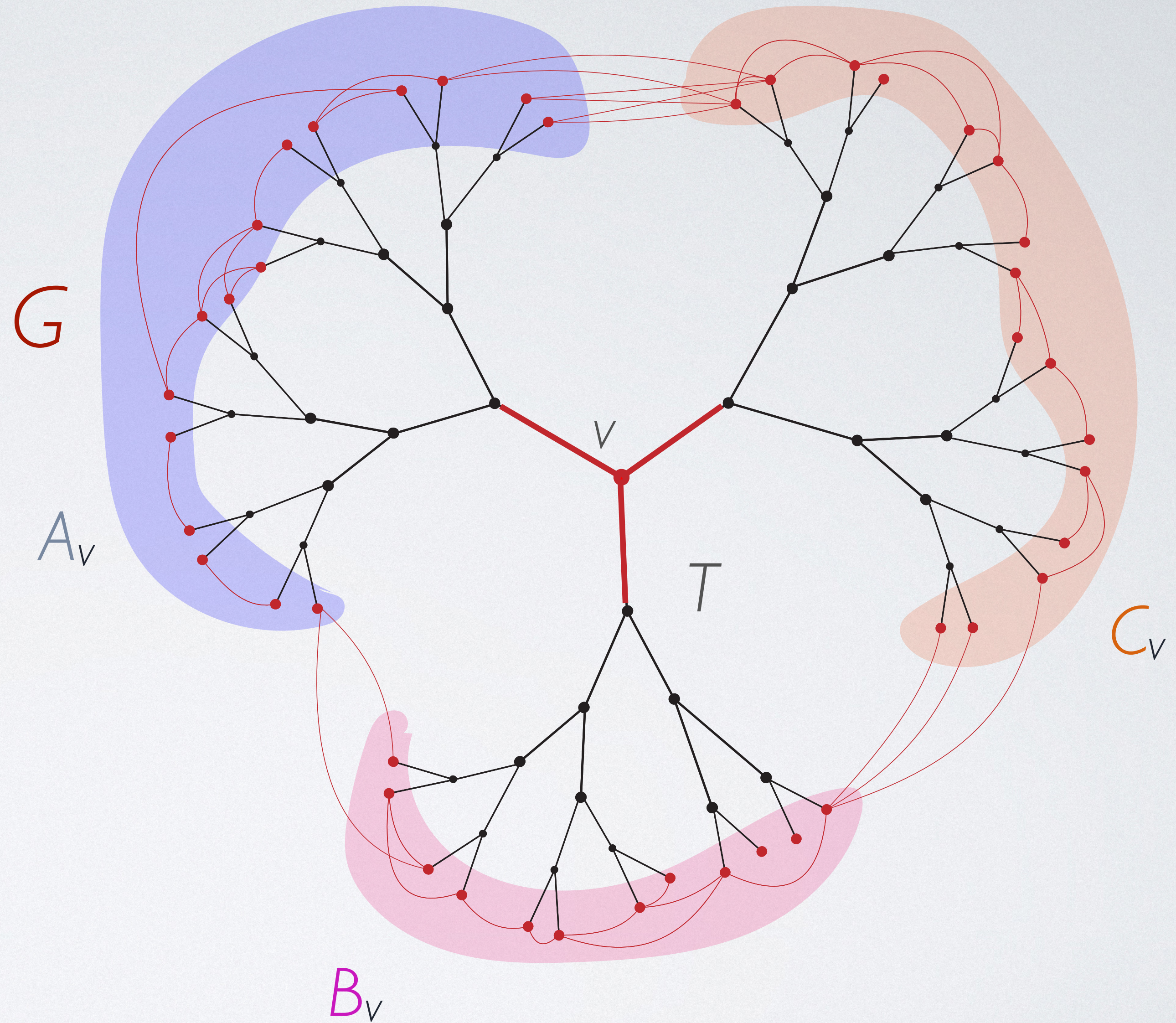
$$\text{flip-width}_\infty(G) \leq O(2^{\text{rank-width}(G)})$$

$$\text{rank-width}(G) \leq k$$



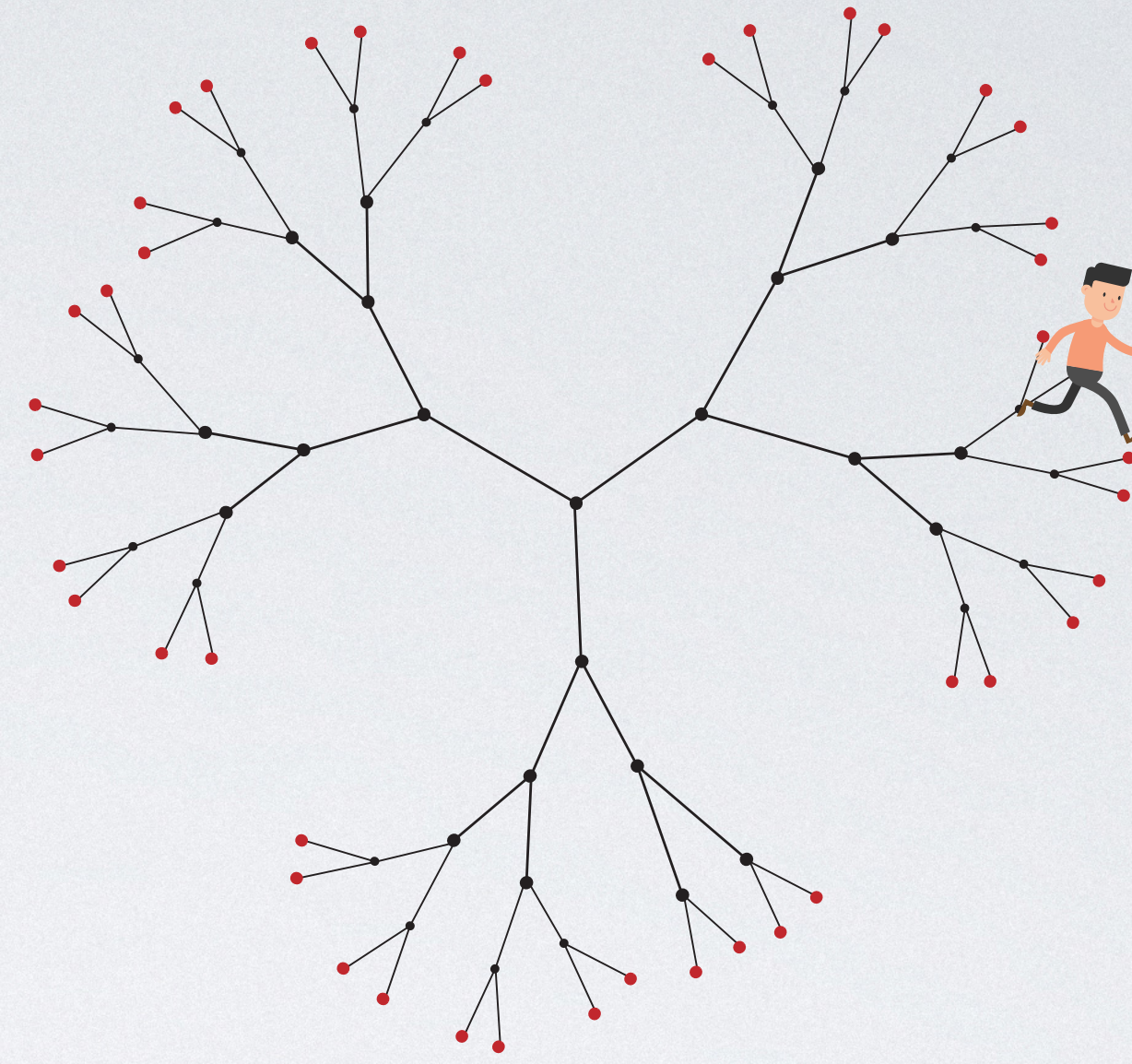
For every node $v \in V(T)$

$\exists O(2^k)$ -flip G' of G
pairwise separating A_v, B_v, C_v

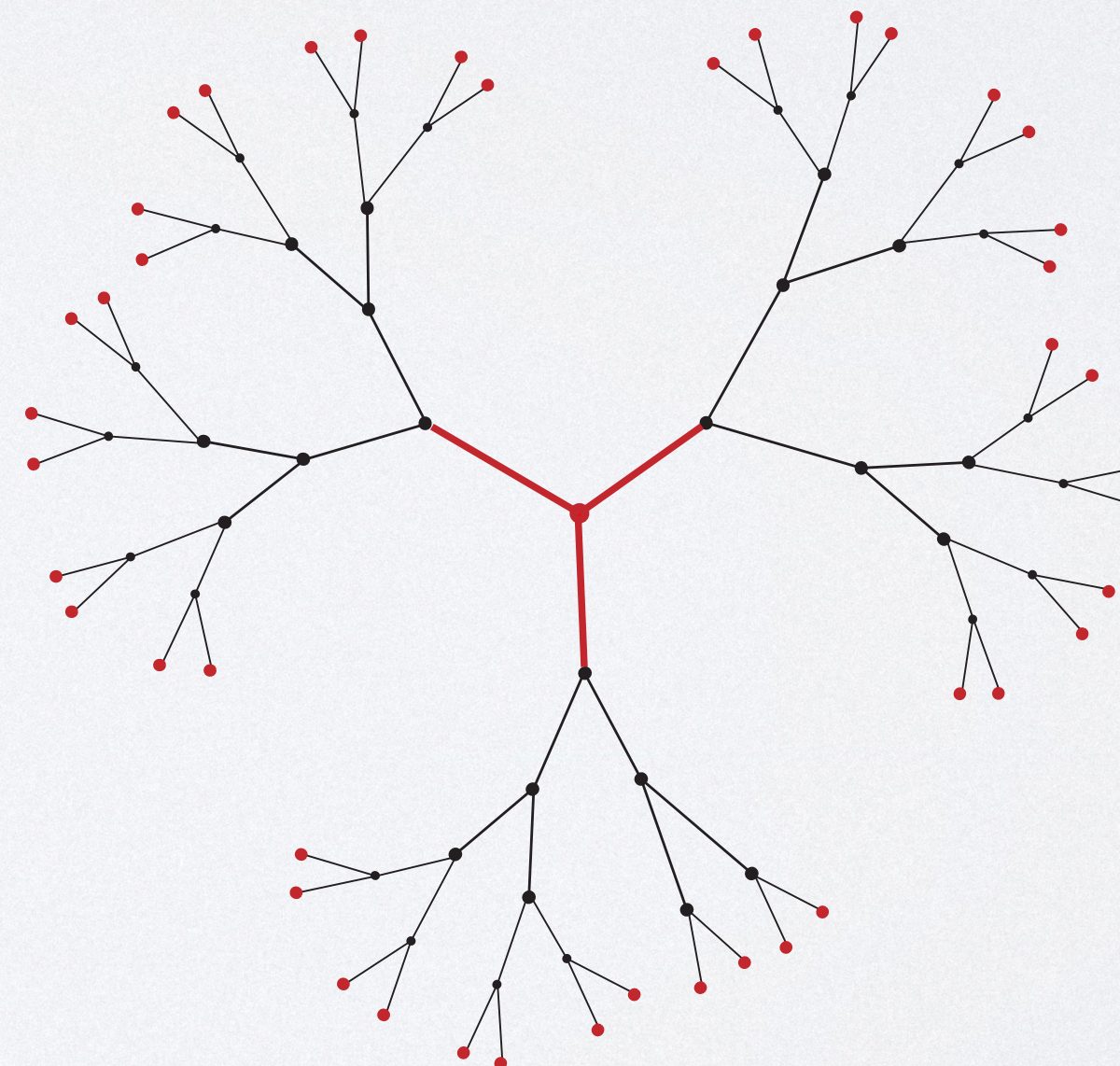


$$\text{flip-width}_\infty(G) \leq O(2^{\text{rank-width}(G)})$$

$$\text{rank-width}(G) \leq k$$

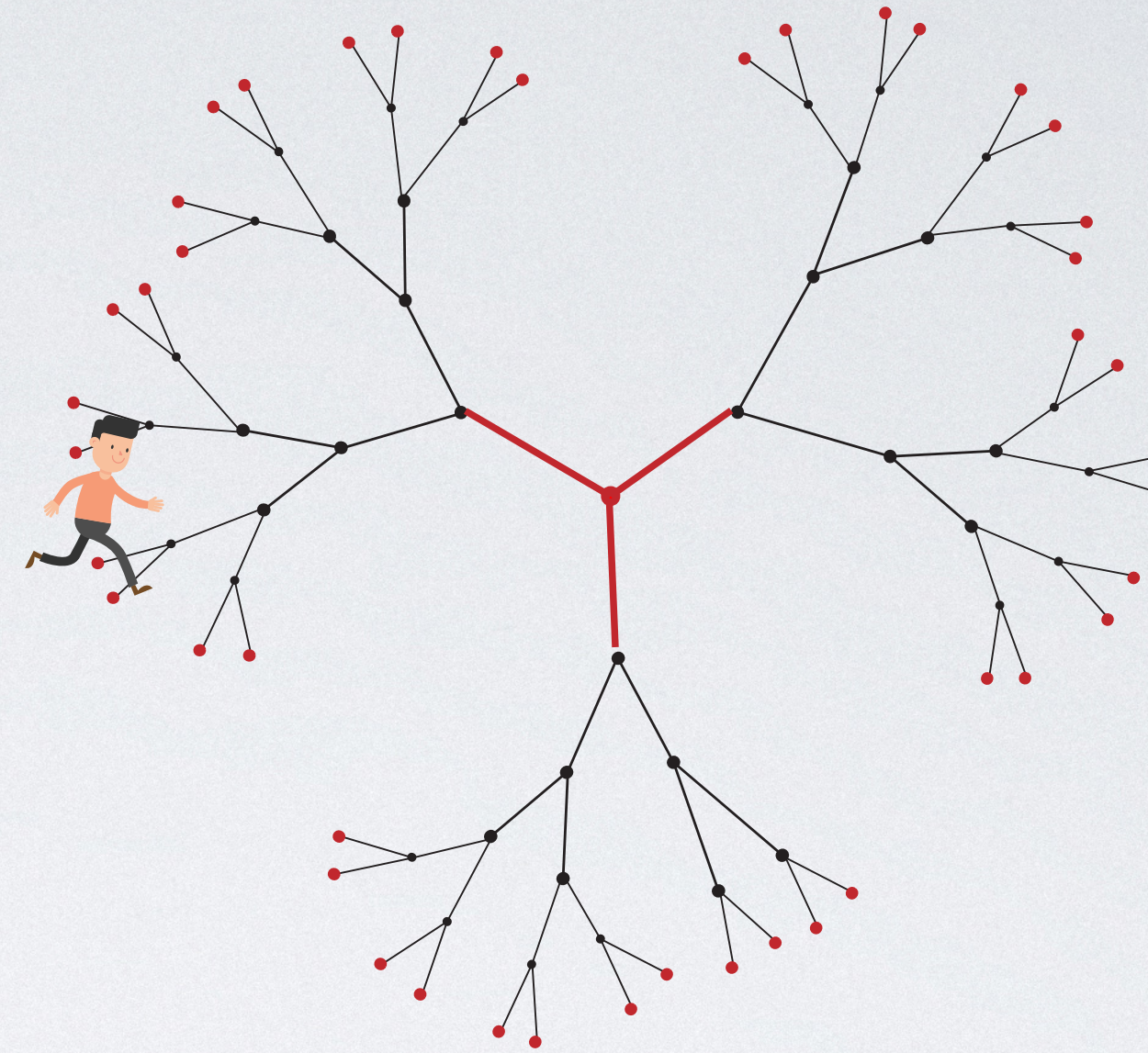


next flip:

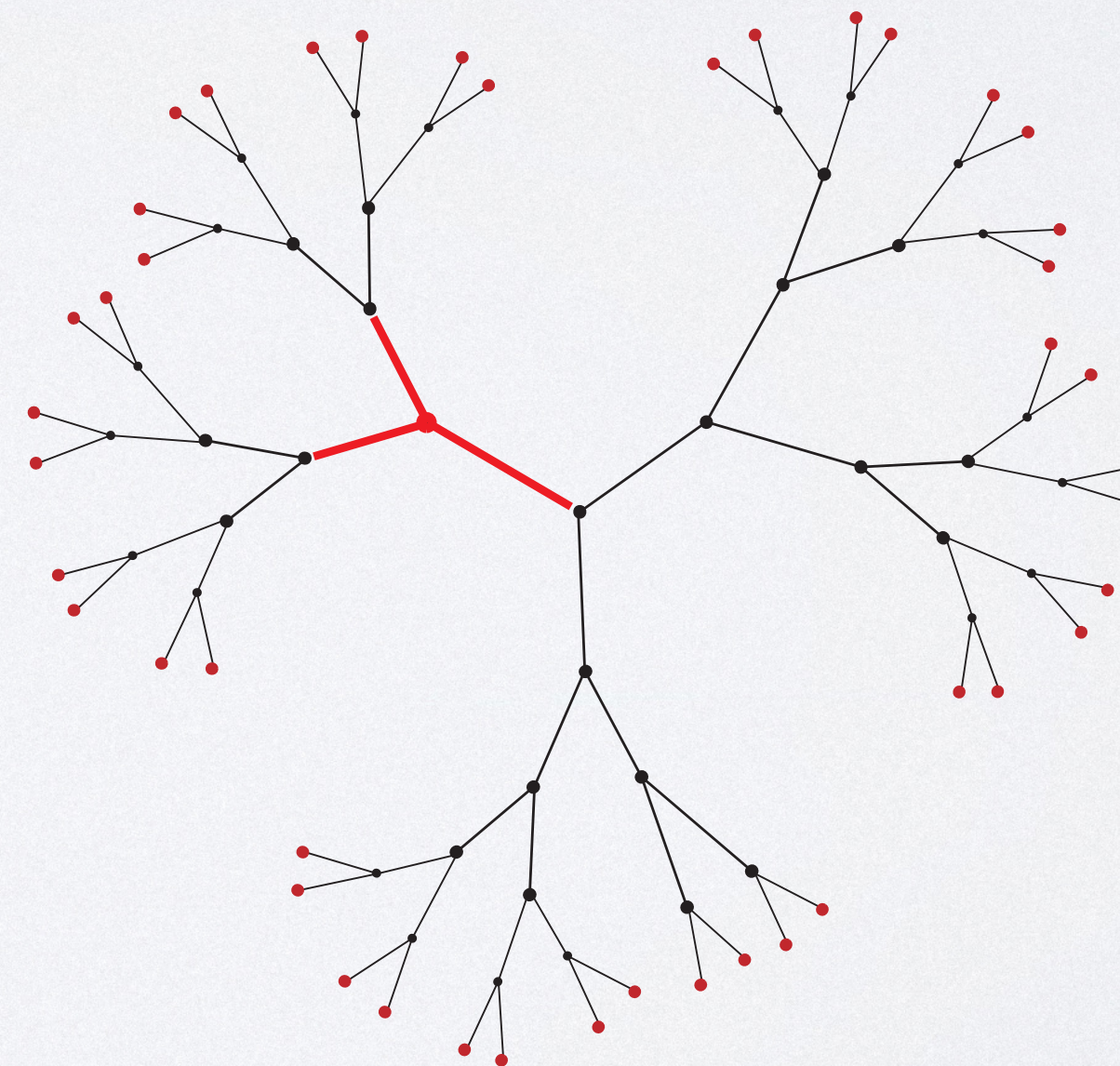


$$\text{flip-width}_\infty(G) \leq O(2^{\text{rank-width}(G)})$$

$$\text{rank-width}(G) \leq k$$



next flip:

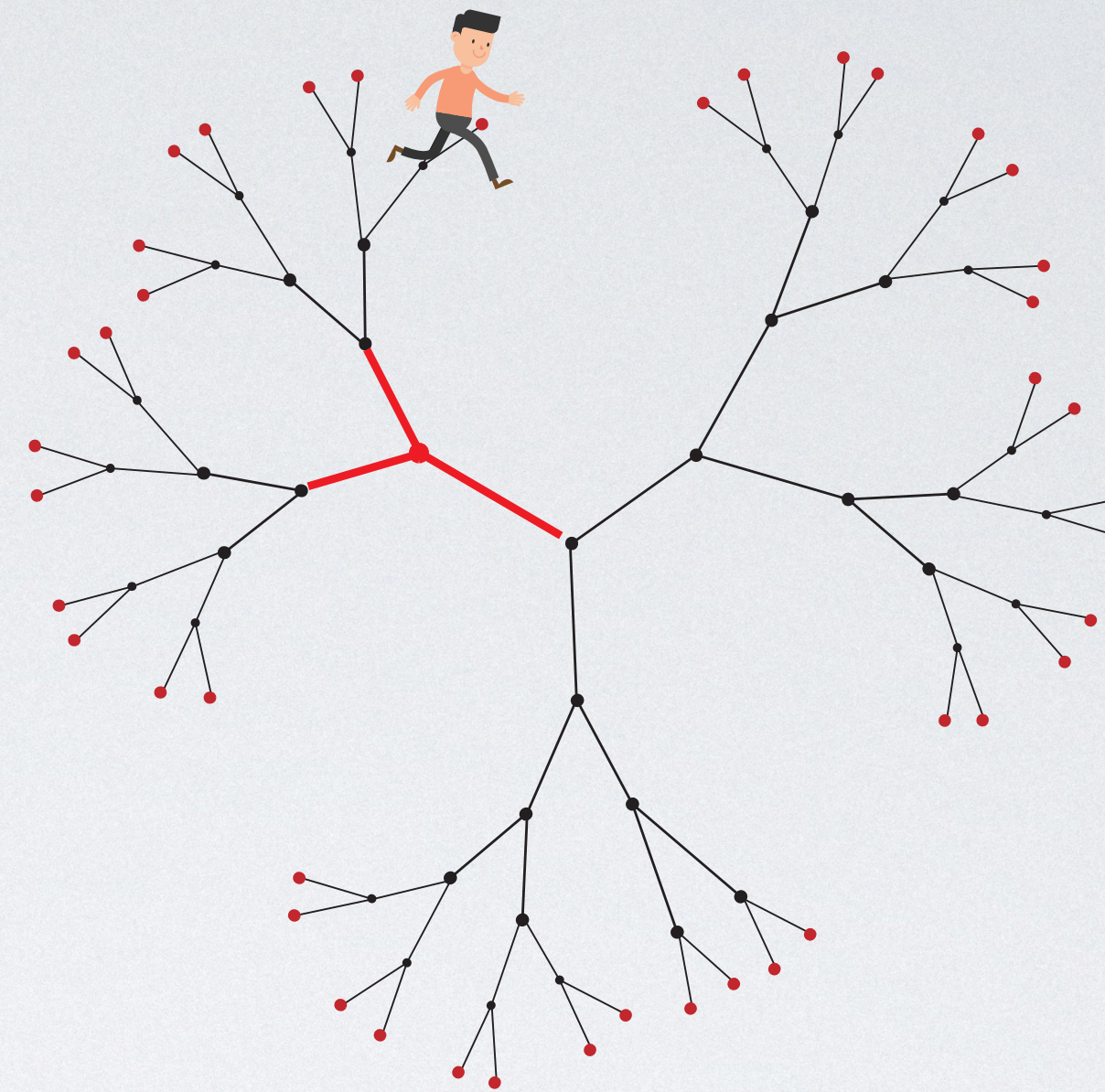


$$\text{flip-width}_\infty(G) \leq O(2^{\text{rank-width}(G)})$$

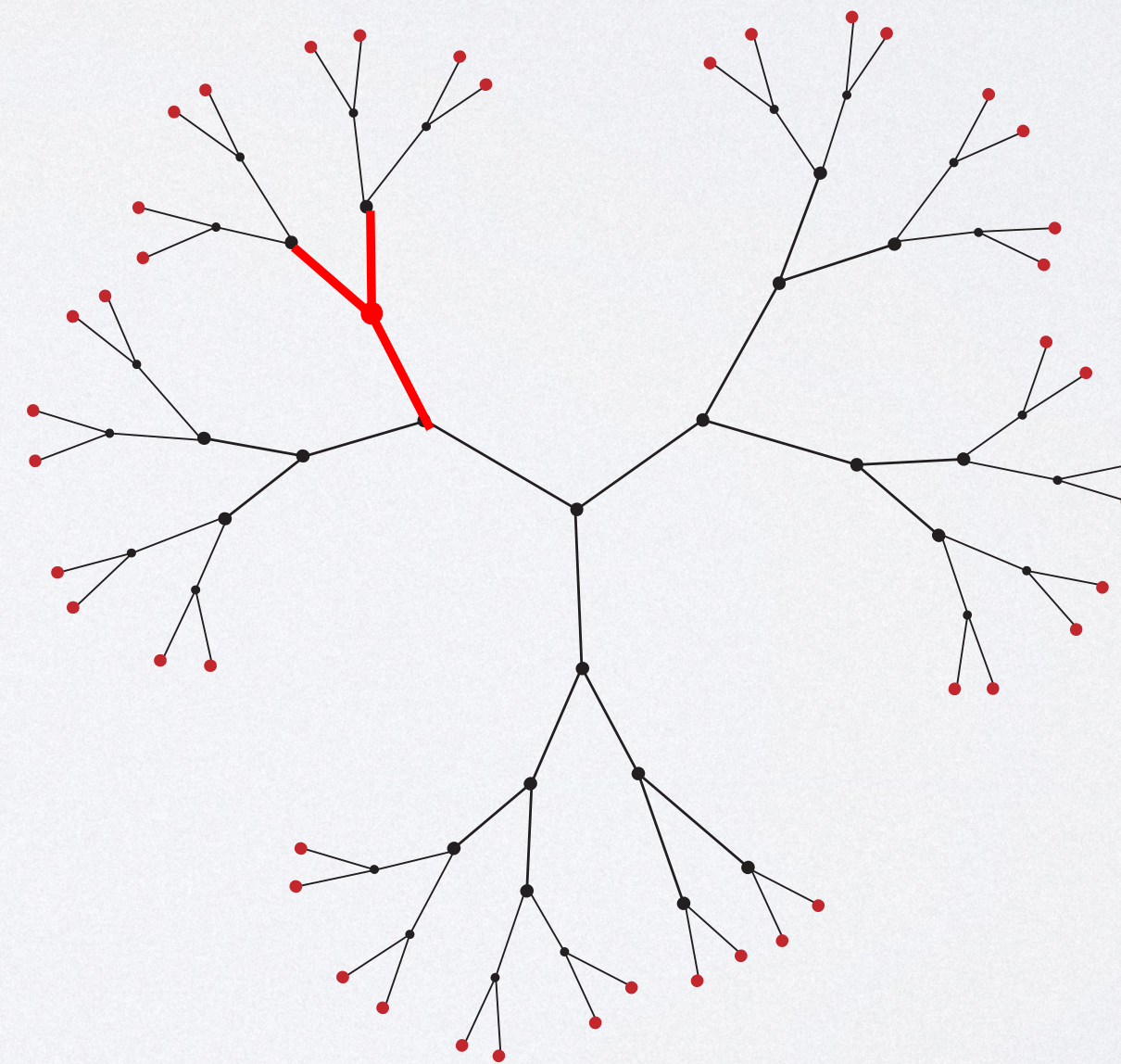
$$\text{rank-width}(G) \leq k$$

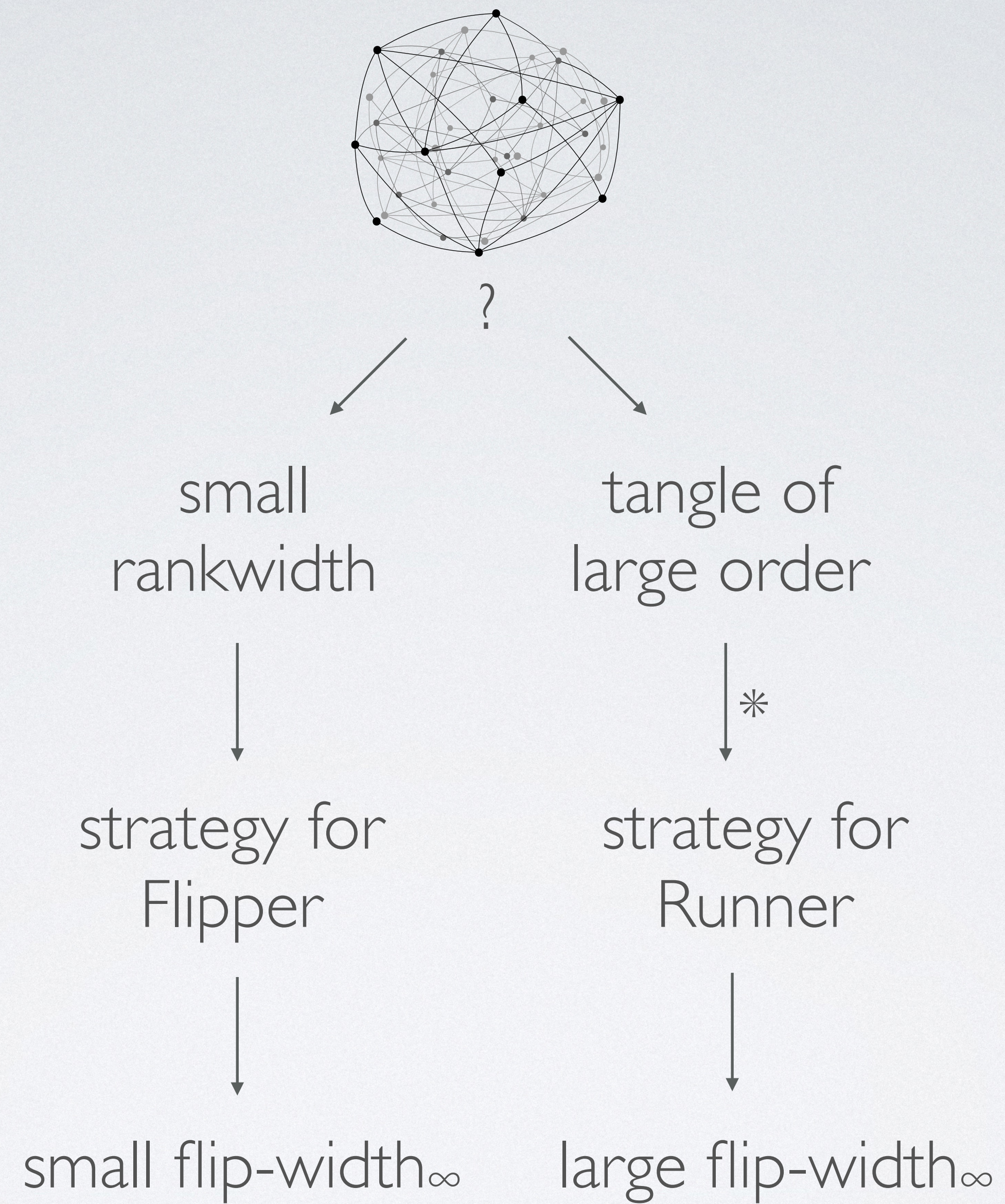


$$\text{flip-width}_\infty(G) \leq O(2^k)$$



next flip:

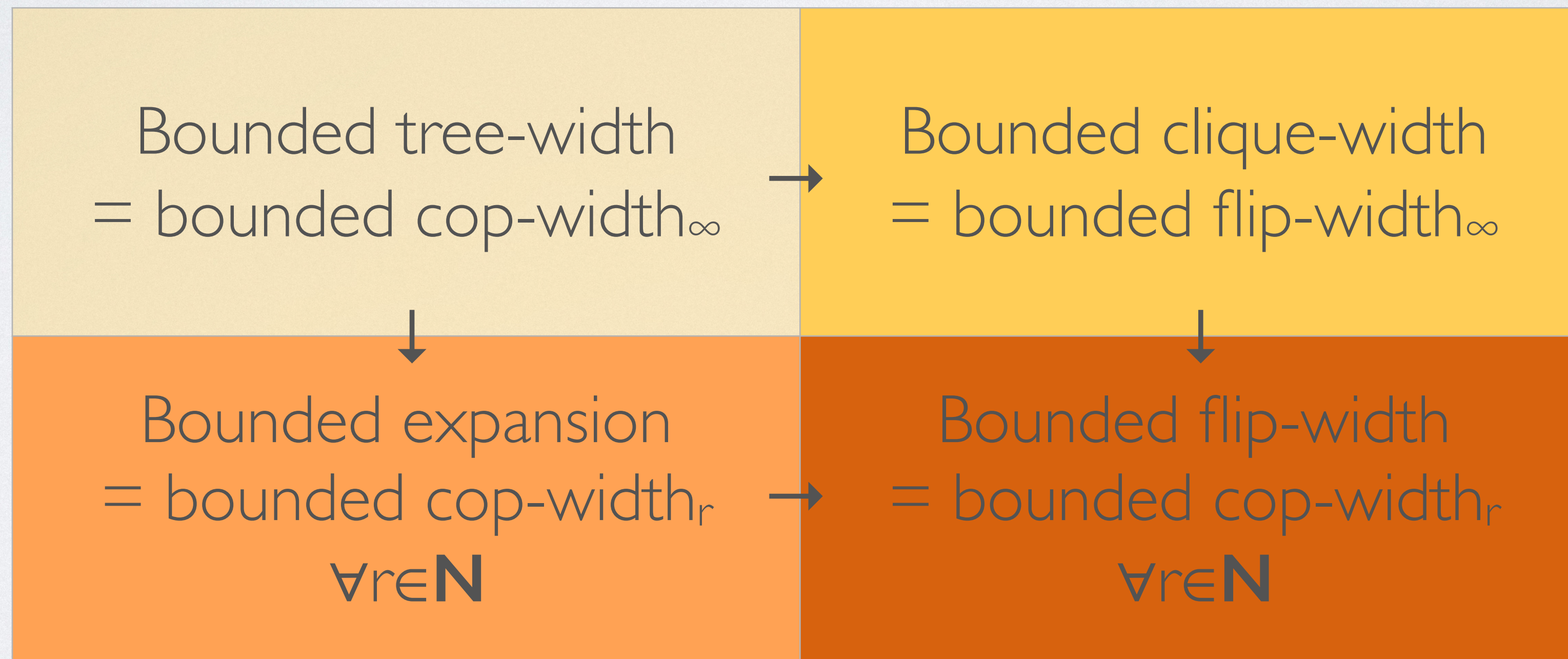




*Oum, private communication

BOUNDED FLIP-WIDTH

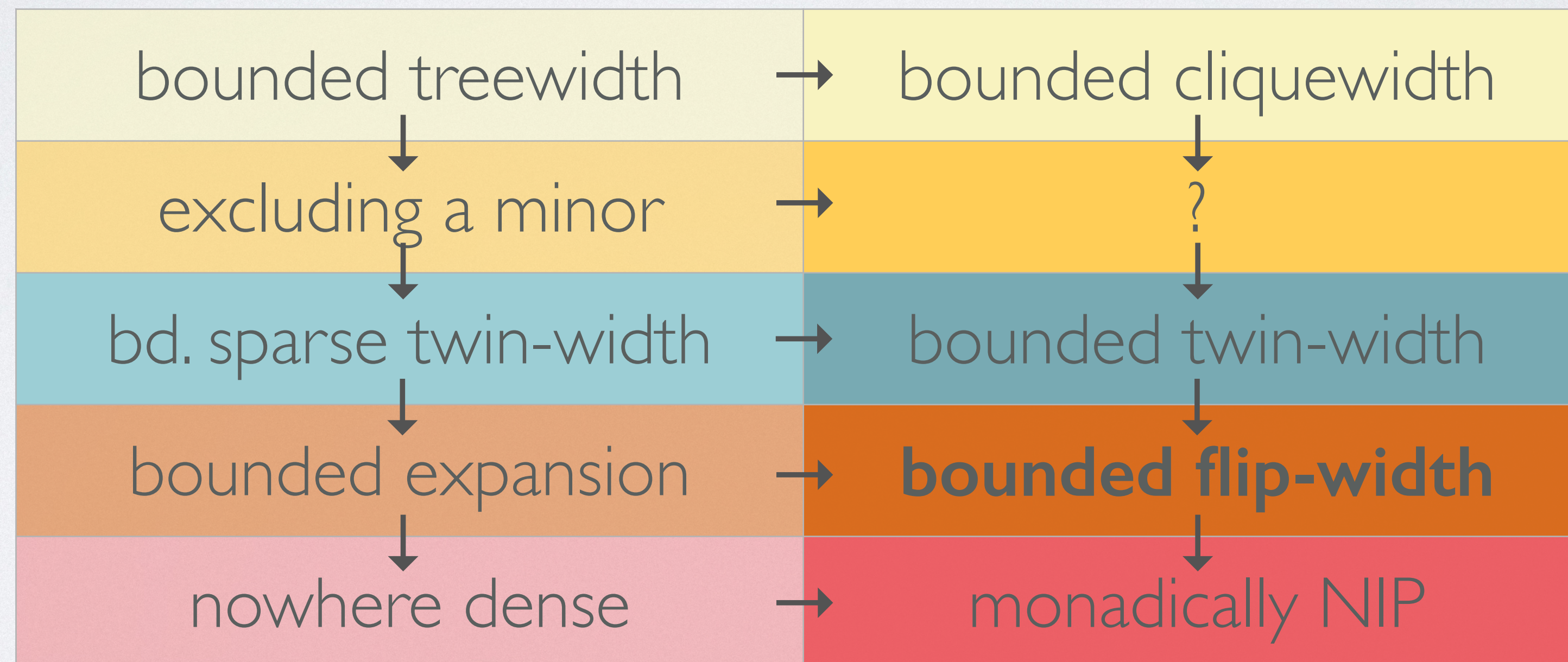
Definition A graph class $\mathcal{C}$ has *bounded flip-width* if
for all $r \in \mathbf{N}$, $\text{flip-width}_r(\mathcal{C}) < \infty$



BOUNDED FLIP-WIDTH

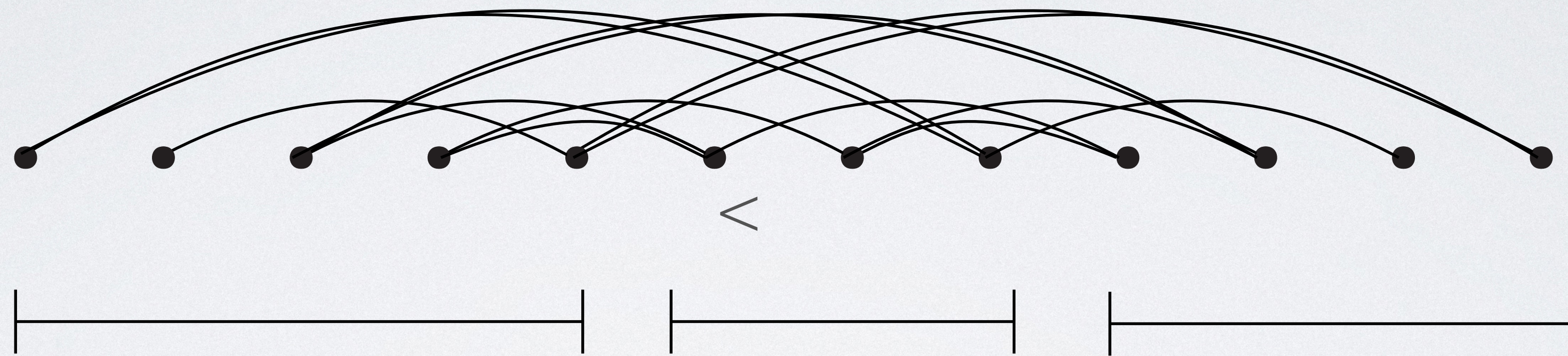
Examples:

- Classes of bounded expansion
- Classes of bounded clique-width
- Classes of bounded twin-width



FLIP-WIDTH OF ORDERED GRAPHS

Variant of flip-width for ordered graphs $G=(V,E,<)$



Flipper performs k -flip on (V,E) and cuts $<$ into k intervals

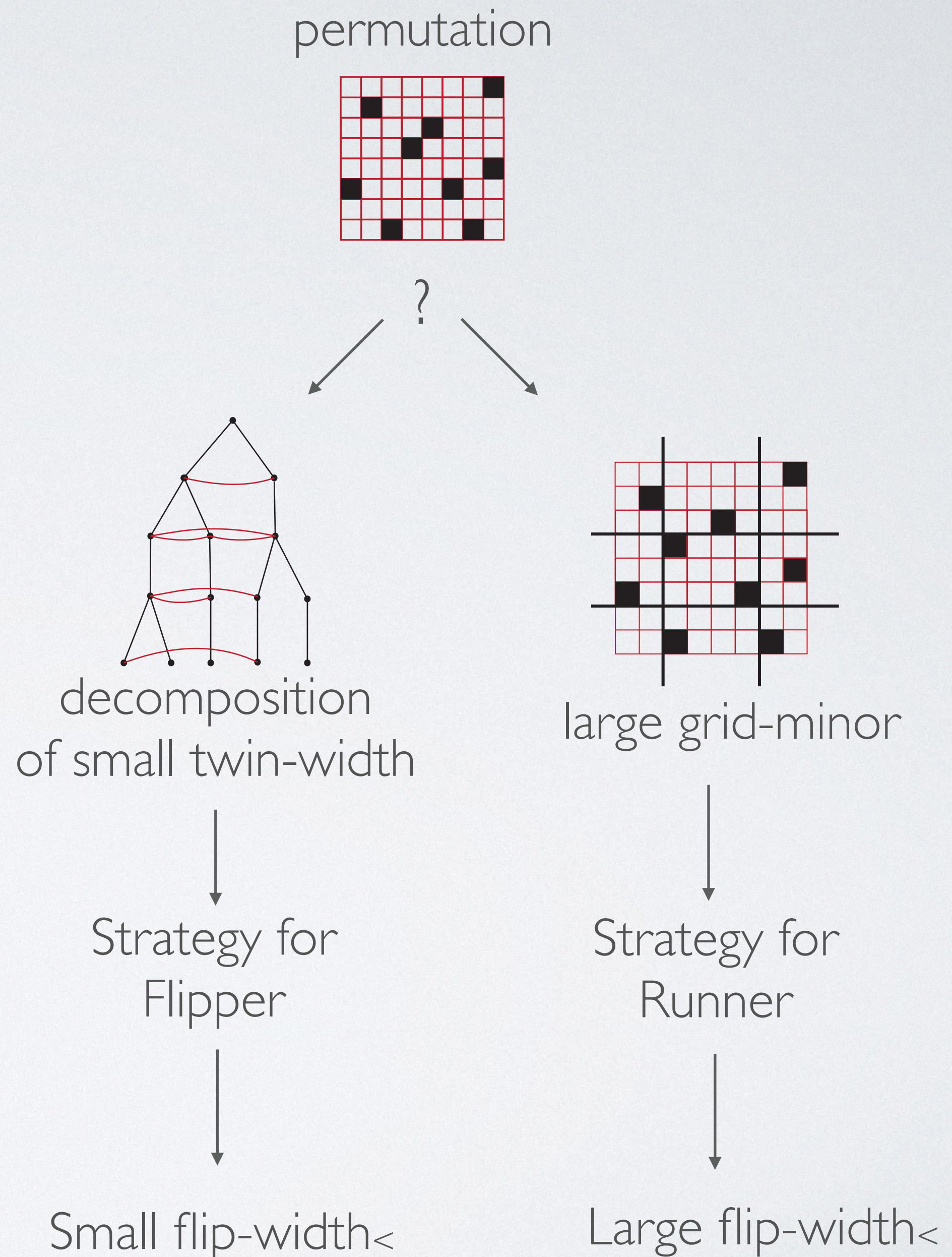
Runner moves along edges at speed 1 or within intervals at speed ∞

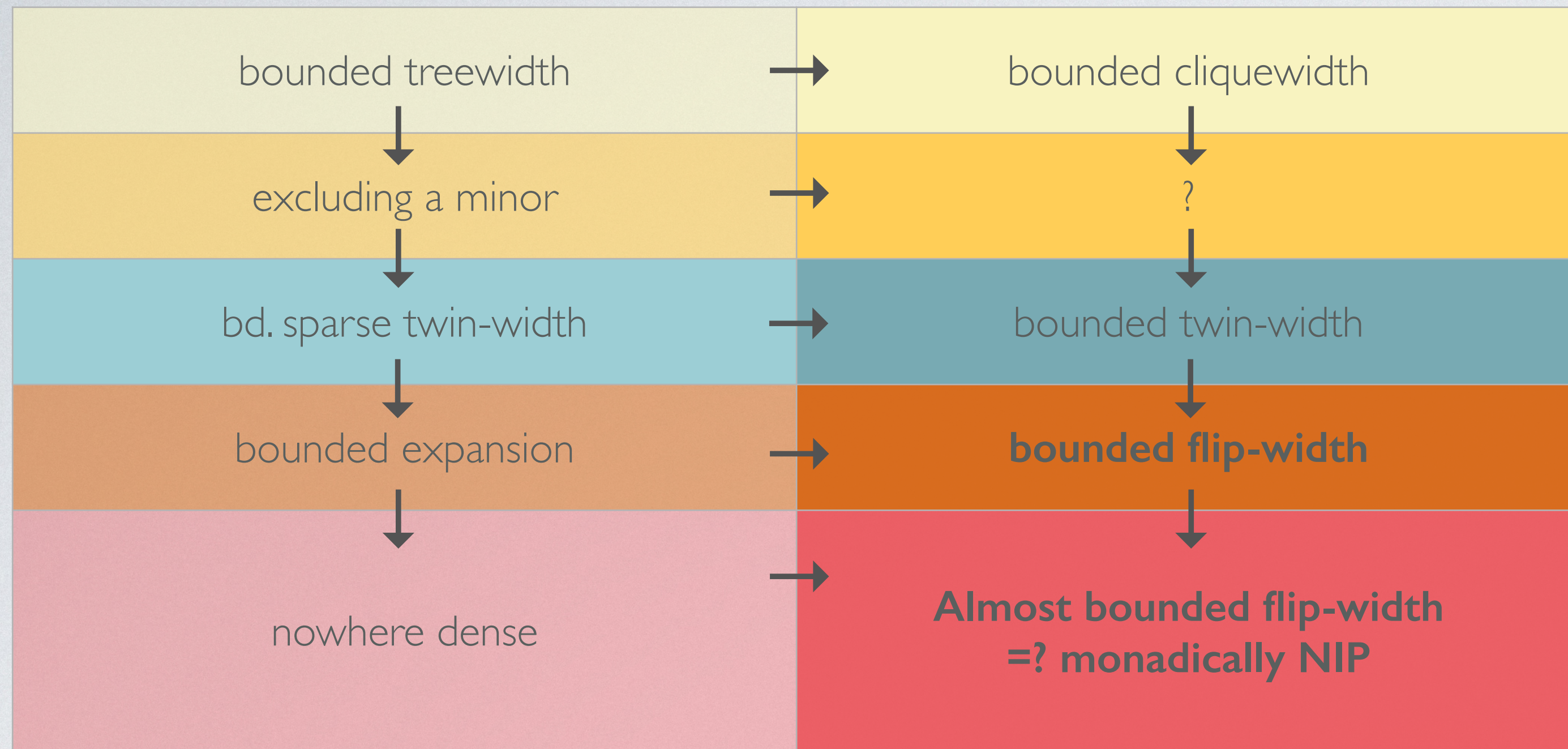
Theorem $\text{flip-width}_{<}(G) \approx \text{twin-width}(G)$

Game characterization of twin-width

TWIN-WIDTH

- Klazar 2000, Marcus&Tardos 2004, Guillemot&Marx 2014 (dichotomy for permutations)
- Bonnet, Kim, Thomassé, Watrigant 2020 (twin-width)
- Bonnet, Giocanti, O. de Mendez, Thomassé, Simon, **T.** 2022 (dichotomy for ordered graphs)





Questions:

- FPT Model checking
- FPT approximation
- Decompositions
- Obstructions
- Dense variant of excluding a minor

THANK YOU !

Looking for students, postdocs – starting from 2024!

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