

Flip-Breakability: Combinatorial Characterizations of Monadically NIP Graph Classes

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The FO Model Checking Problem

Problem: Given a graph G and an FO sentence φ , decide whether

$$G \models \varphi.$$

Example: G contains a dominating set of size k iff.

$$G \models \exists x_1 \dots \exists x_k \forall y : \bigvee_{i \in [k]} (y = x_i \vee y \sim x_i).$$

Runtime: On the class of all graphs, FO model checking is $AW[*]$ -hard. We will assume $FPT \neq AW[*]$.

Question: On which classes is FO model checking fixed-parameter tractable, i.e., solvable in time $f(\varphi) \cdot n^c$?

Tractable Classes

Theorem [Grohe, Kreutzer, Siebertz, 2014]

Let $\mathcal{C}$ be a **monotone** class of graphs.

$\mathcal{C}$ admits fpt FO model checking if and only if $\mathcal{C}$ is **nowhere dense**.

Theorem [Dreier, Eleftheriadis, Mählmann, McCarty, Pilipczuk, Toruńczyk, 2023+]

Let $\mathcal{C}$ be a **hereditary and orderless** class of graphs.

$\mathcal{C}$ admits fpt FO model checking if and only if $\mathcal{C}$ is **monadically stable**.

Theorem [Bonnet, Giocanti, Ossona de Mendez, Simon, Thomassé, Toruńczyk, 2022]

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A Unifying Theory

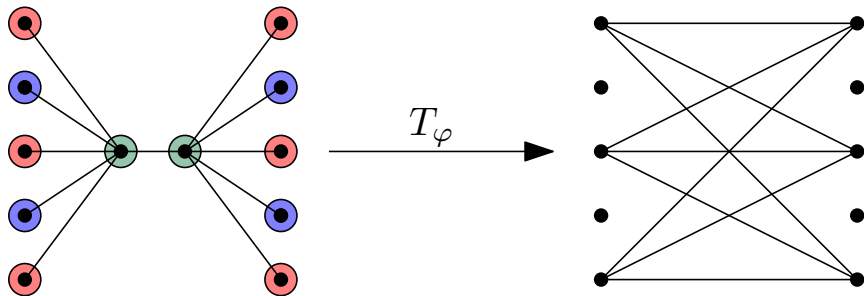
Conjecture

*Let $\mathcal{C}$ be a **hereditary** class of graphs.*

*$\mathcal{C}$ admits fpt FO model checking if and only if $\mathcal{C}$ is **monadically NIP**.*

FO Transductions

Transductions $\hat{=}$ coloring + interpreting + taking an induced subgraph

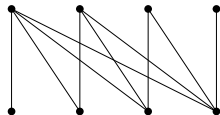


$$\varphi(x, y) := \text{Red}(x) \wedge \text{Red}(y) \wedge \text{dist}(x, y) = 3$$

Monadic Stability and Monadic NIP

Definition

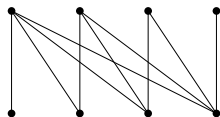
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Monadic Stability and Monadic NIP

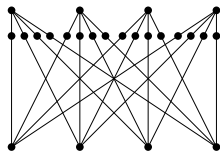
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Definition

A class is *monadically NIP*, if it does not transduce the class of *all graphs*.
Equivalently, it does not transduce the class of all 1-subdivided bicliques.



Wanted: Combinatorial Characterizations

Monadically NIP classes are defined using **logic**.

Working towards algorithms we need tools that are **combinatorial**.

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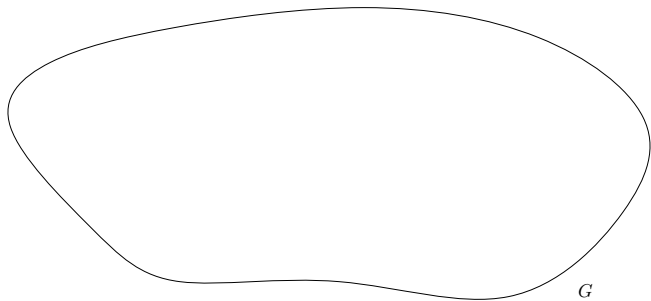
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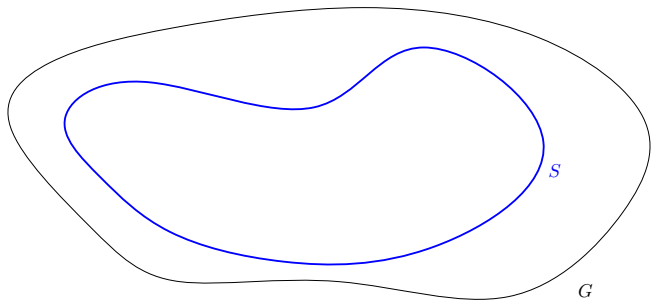
In this talk we will present:

- a combinatorial structure characterization: **flip-breakability**
- a combinatorial non-structure characterization: **forbidden induced subgraphs**

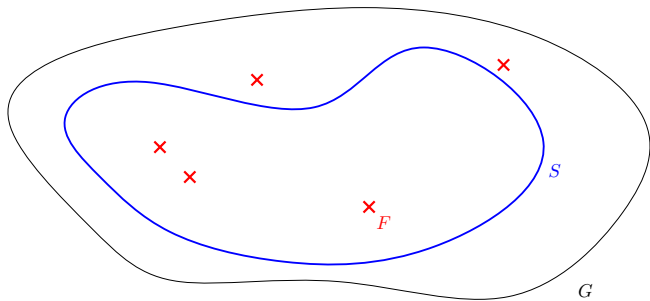
Characterizing Nowhere Denseness: Uniform Quasi-Wideness



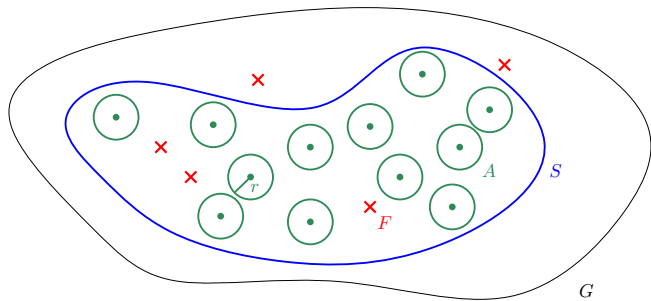
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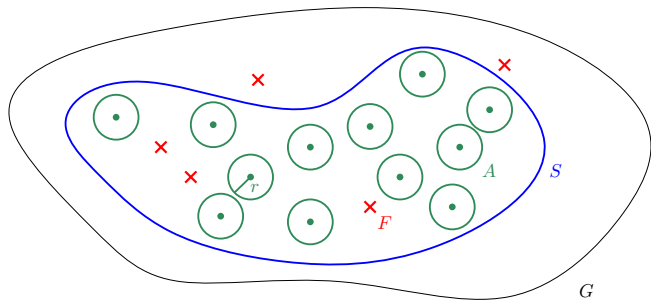
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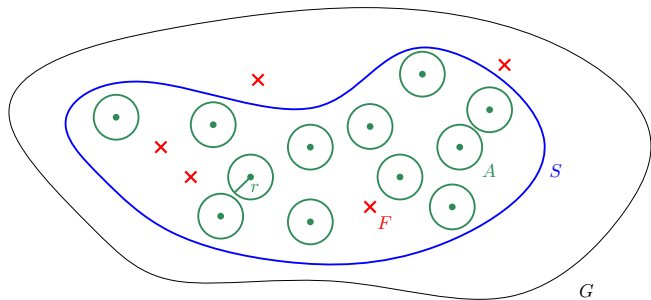
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Uniform Quasi-Wideness (slightly informal)

A class $\mathcal{C}$ is *uniformly quasi-wide* if for every radius r , in every large set S we find a still large set A that is r -independent after removing a set F of constantly many vertices.

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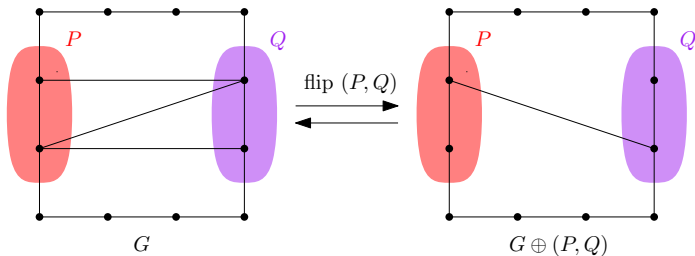
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Theorem [Něsetřil, Ossona de Mendez, 2011]

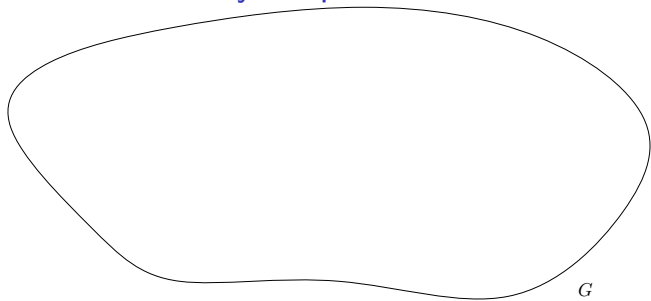
A class $\mathcal{C}$ is uniformly quasi-wide if and only if it is nowhere dense.

Towards Dense Graphs

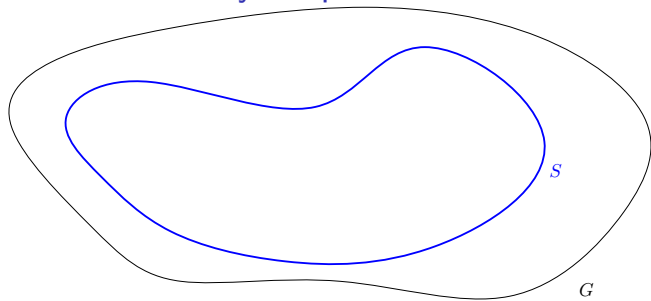
Denote by $G \oplus (P, Q)$ the graph obtained from G by complementing edges between pairs of vertices from $P \times Q$.



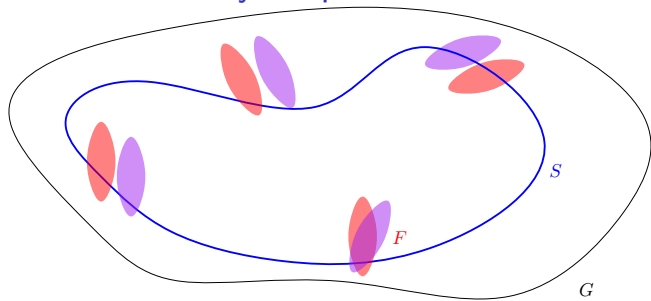
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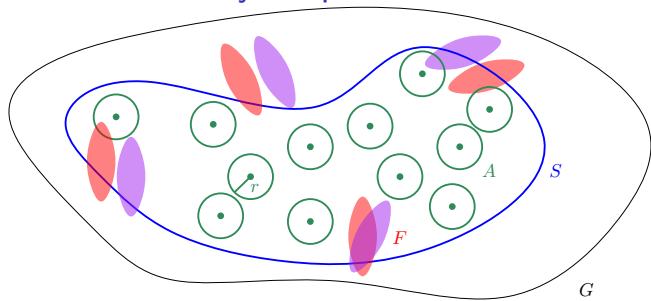
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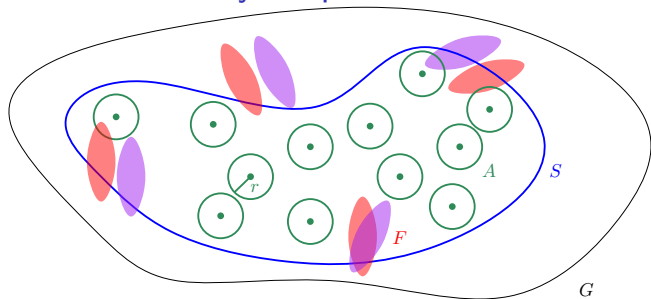
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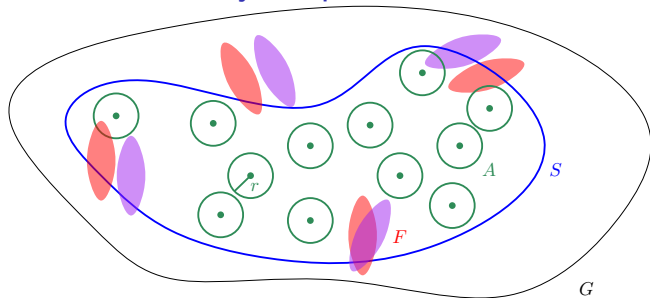
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A class $\mathcal{C}$ is *flip-flat* if for every radius r , in every large set S we find a still large set A that is r -independent after performing a set F of constantly many flips.

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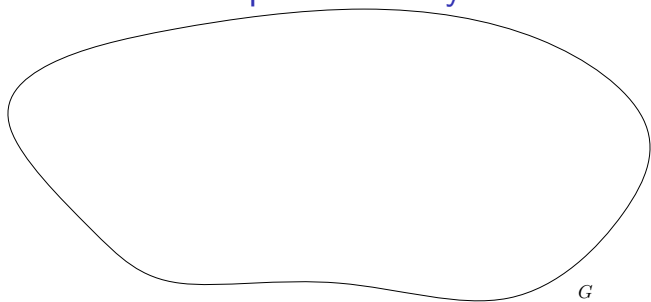
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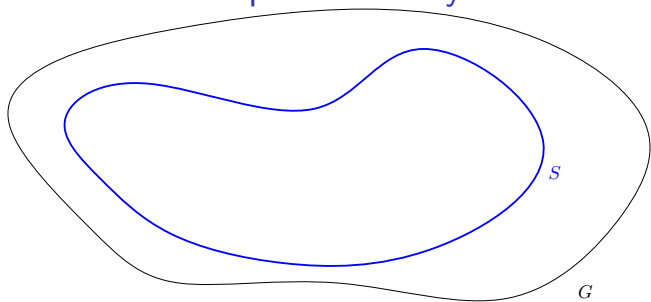
Theorem [Dreier, Mählmann, Siebertz, Toruńczyk, 2022]

A class $\mathcal{C}$ is flip-flat if and only if it is monadically stable.

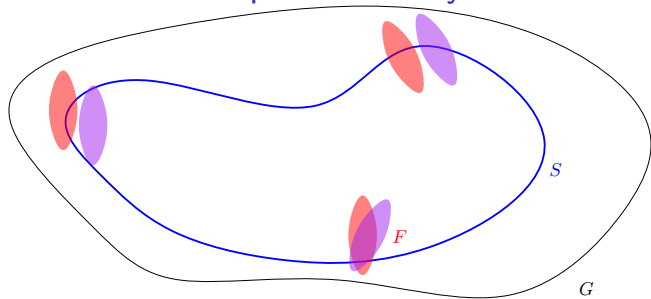
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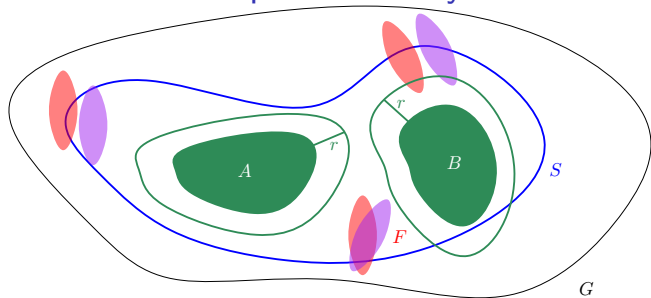
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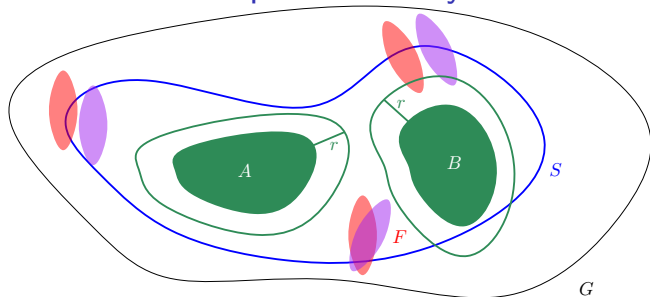
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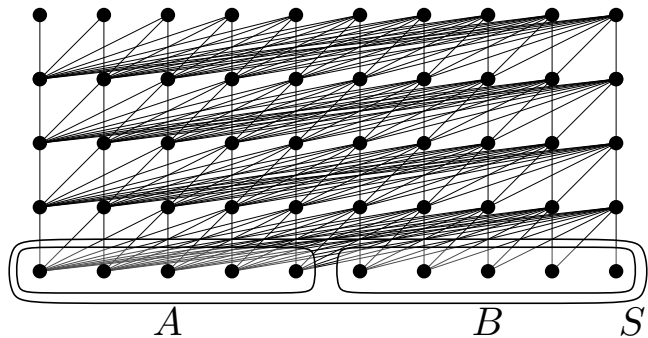
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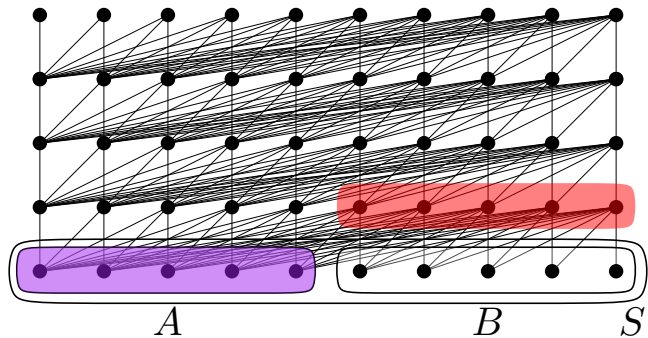
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A class $\mathcal{C}$ is *flip-breakable* if for every radius r , in every large set S we find two large sets A and B that and a flip F of bounded size such that $N_{G \oplus F}^r(A) \cap N_{G \oplus F}^r(B) = \emptyset$.

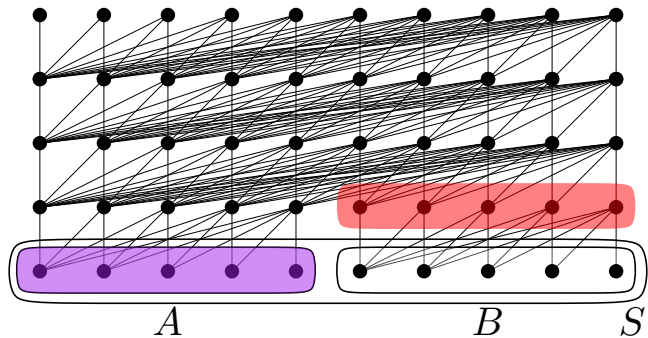
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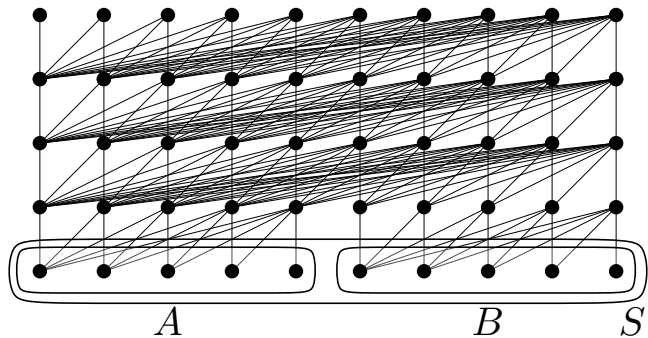
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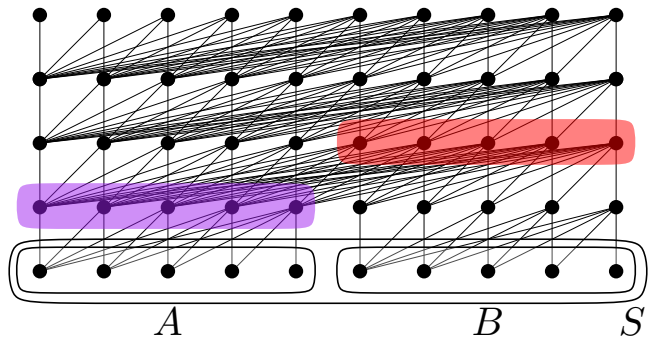
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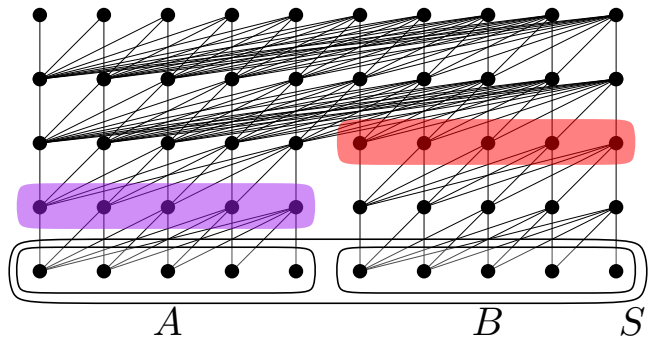
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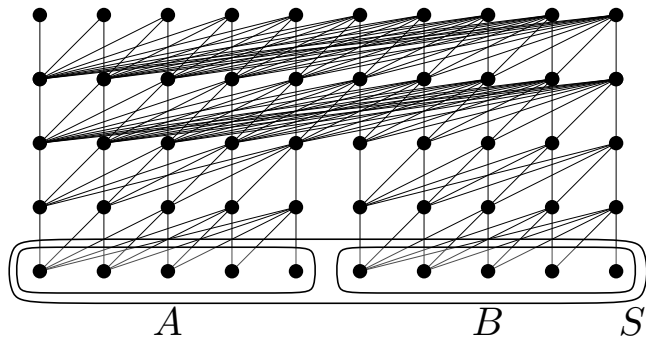
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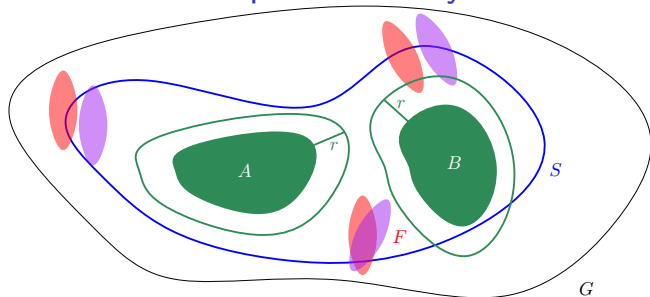
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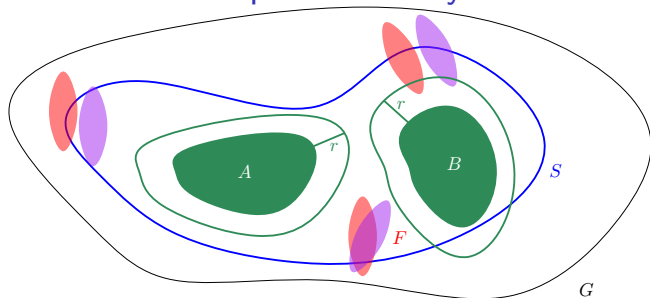
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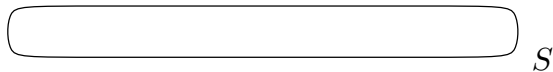
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Theorem [Dreier, Mählmann, Toruńczyk]

A class $\mathcal{C}$ is flip-breakable if and only if it is monadically NIP.

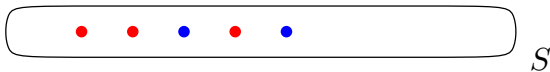
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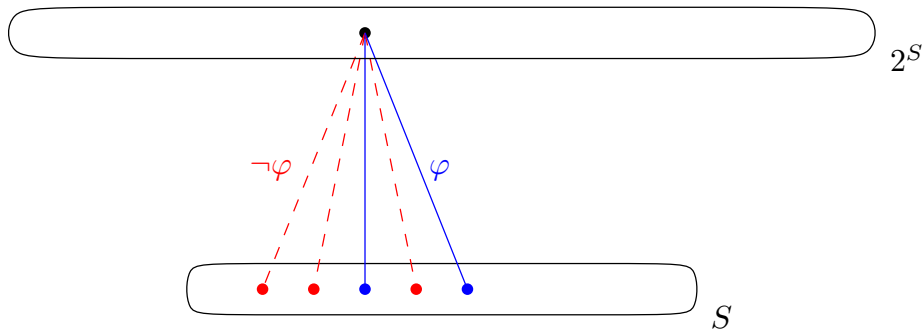
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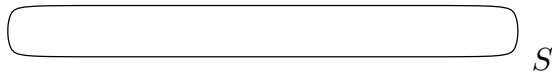
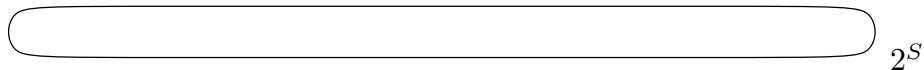
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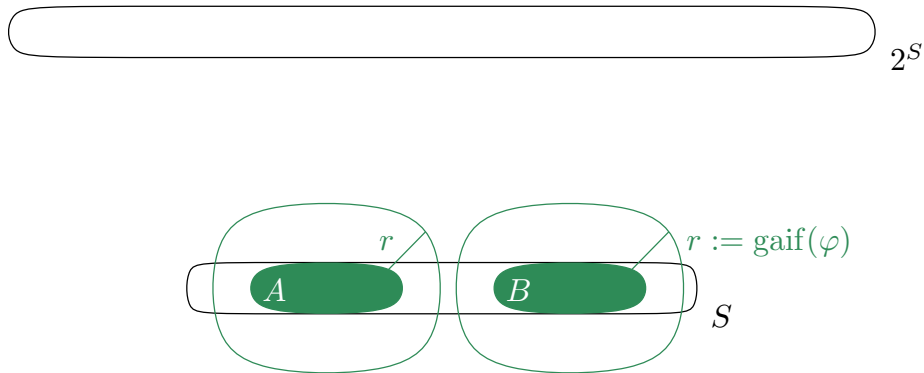
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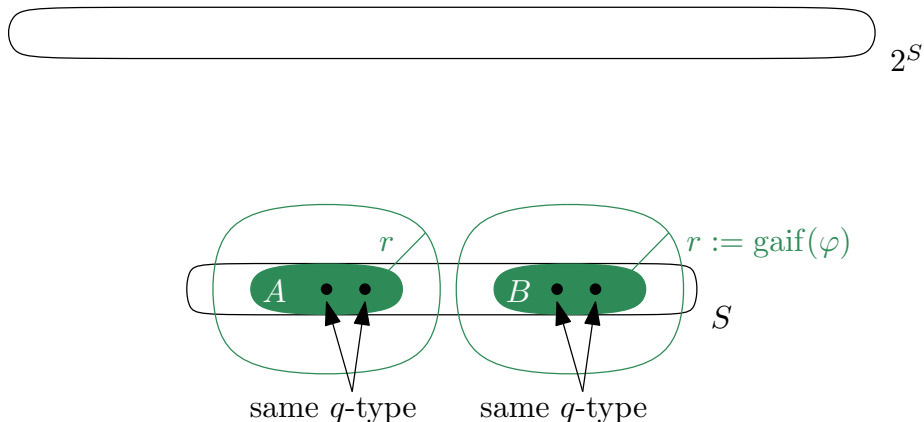
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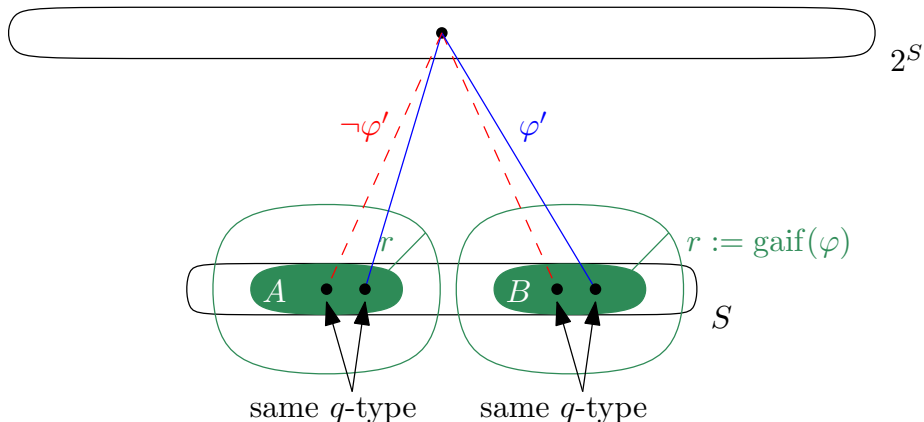
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dist- r	flip-	monadic stability	monadic NIP
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dist- ∞	flip-		
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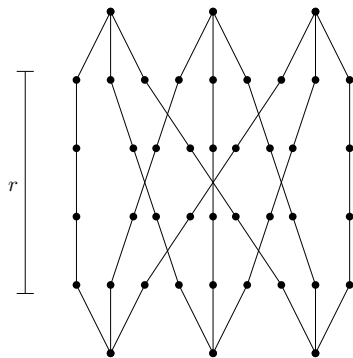
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	deletion-	bd. treedepth	bd. treewidth

Wanted: Combinatorial Characterizations

In this talk we will present:

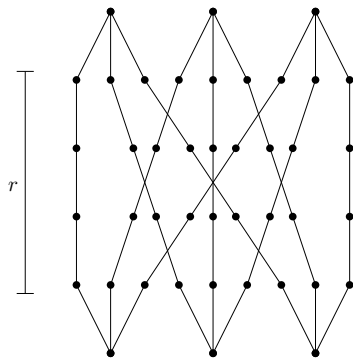
- a combinatorial structure characterization: flip-breakability ✓
- a combinatorial non-structure characterization: forbidden induced subgraphs

Characterizing Monadic NIP by Forbidden Induced Subgraphs

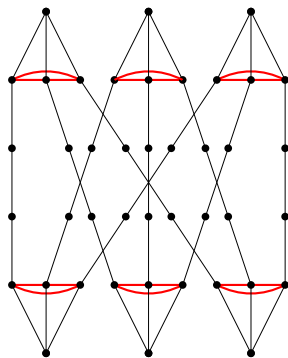


star r -crossing
 $= r$ -subdivided biclique

Characterizing Monadic NIP by Forbidden Induced Subgraphs

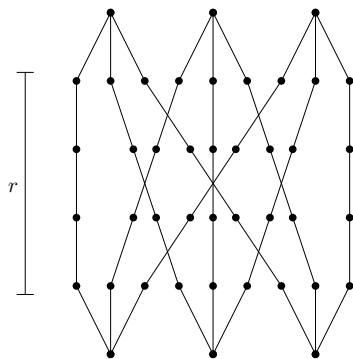


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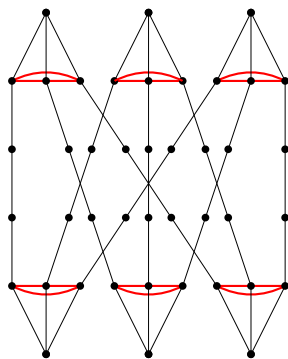


clique r -crossing

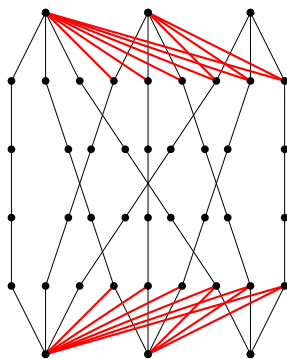
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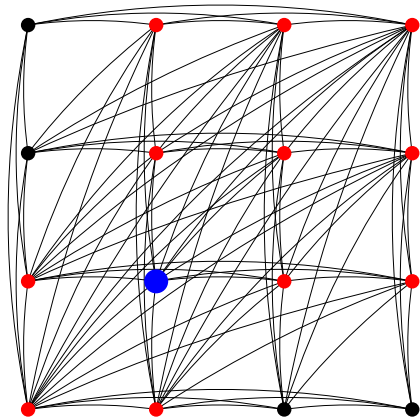


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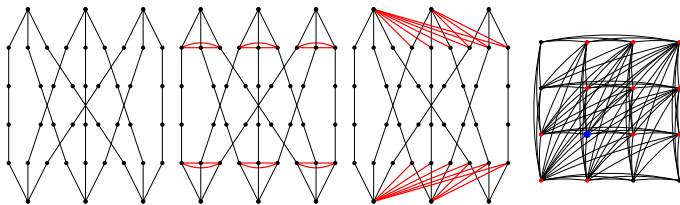
half-graph r -crossing

Characterizing Monadic NIP by Forbidden Induced Subgraphs



comparability grid

Characterizing Monadic NIP by Forbidden Induced Subgraphs

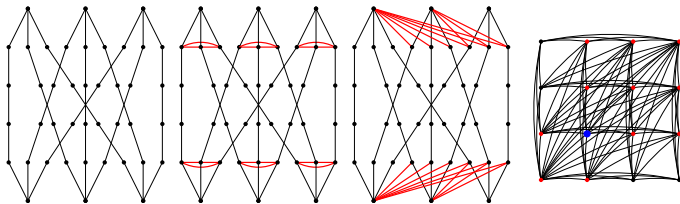


Theorem [Dreier, Mählmann, Toruńczyk]

Let $\mathcal{C}$ be a graph class. Then $\mathcal{C}$ is monadically NIP if and only if for every $r \geq 1$ there exists $k \in \mathbb{N}$ such $\mathcal{C}$ excludes as induced subgraphs

- all layerwise **flipped star r -crossings** of order k , and
- all layerwise **flipped clique r -crossings** of order k , and
- all layerwise **flipped half-graph r -crossings** of order k , and
- **the comparability grid** of order k .

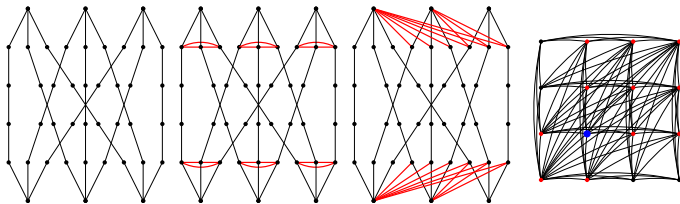
Forbidden Induced Subgraphs: Applications



Theorem [Dreier, Mählmann, Toruńczyk]

1. FO Model checking is $AW[*]$ -hard on every hereditary class that is not mon. NIP.

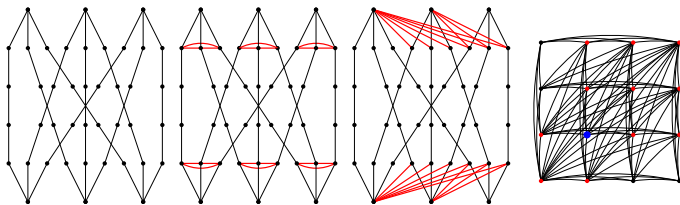
Forbidden Induced Subgraphs: Applications



Theorem [Dreier, Mählmann, Toruńczyk]

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2. Every small hereditary class is monadically NIP.

Forbidden Induced Subgraphs: Applications

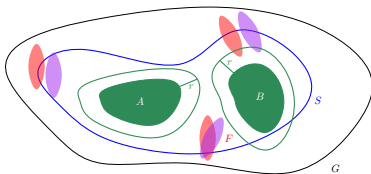


Theorem [Dreier, Mählmann, Toruńczyk]

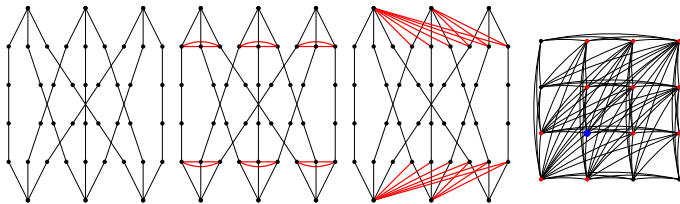
1. FO Model checking is $AW[*]$ -hard on every hereditary class that is not mon. NIP.
2. Every small hereditary class is monadically NIP.
3. Every class with almost bounded flip-width is monadically NIP.

Summary: We give two combinatorial characterizations of mon. NIP graph classes.

A structure characterization called flip-breakability:



A non-structure characterization by forbidden induced subgraphs:



FO model checking is $AW[*]$ -hard on hereditary graph classes that are not mon. NIP.