Canonical decompositions in bounded treedepth and bounded shrubdepth graphs

Wojciech Przybyszewski
Joint work with Pierre Ohlmann, Michał Pilipczuk, Szymon Toruńczyk

LoGAlg 2023

Treedepth game

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1. Splitter chooses a vertex v to delete
2. Connector chooses G_{i+1} as a connected component in $G_i - v$.

Splitter wins once G_i has size 1.

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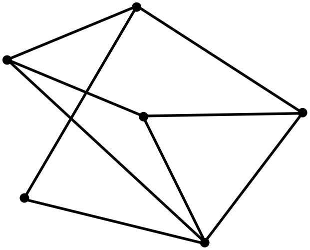

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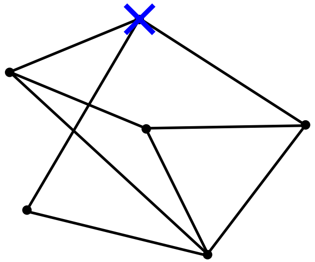

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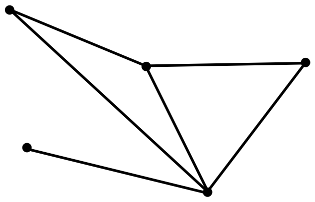

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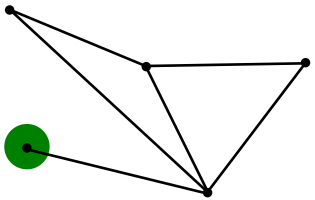

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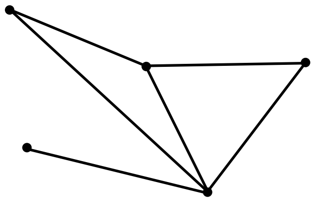

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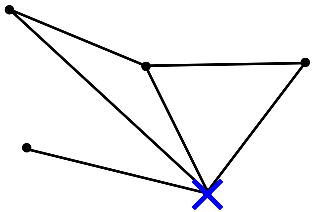

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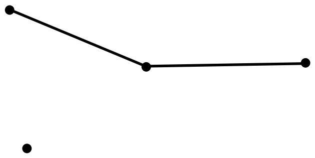

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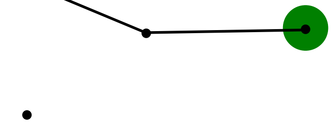

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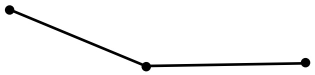

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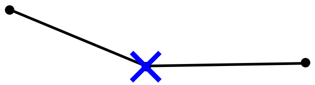

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Observation: We don't need to assume that G is a finite graph for this definition to make sense.

Progressing moves in the treedepth game

Theorem.

There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if a graph G has treedepth d then Splitter has at most $f(d)$ progressing moves[1].

[1] A vertex v is a progressing move for Splitter if every connected component C of $G - \{v\}$ has strictly smaller treedepth than G.

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- $v_\infty \neq v_i$ for every $i \in I$;
- v_∞ is a progressing move.

□

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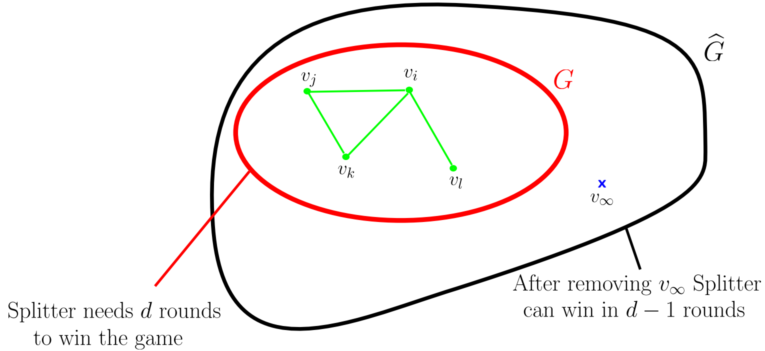

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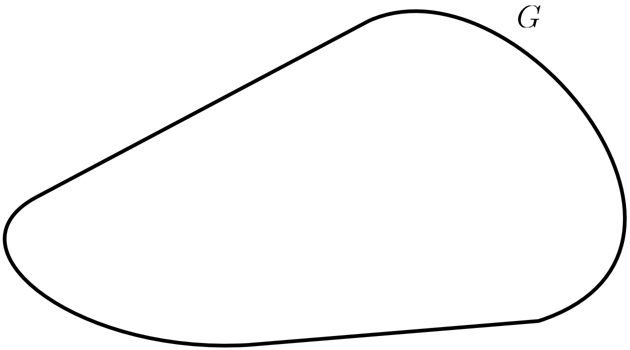
Compactness yields a model that contradicts the previous lemma. □

Canonical decomoposition of graphs of bounded treedepth



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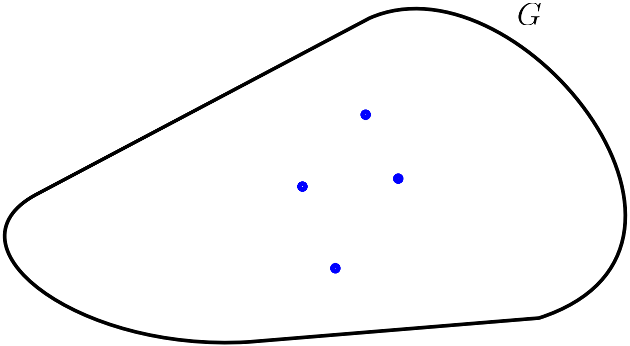

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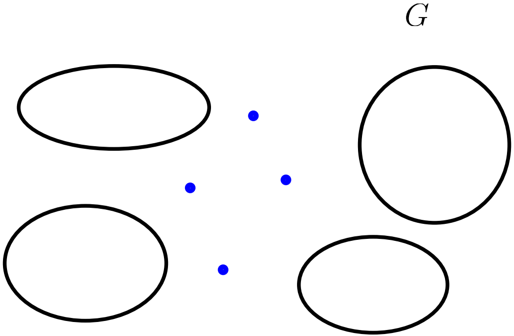

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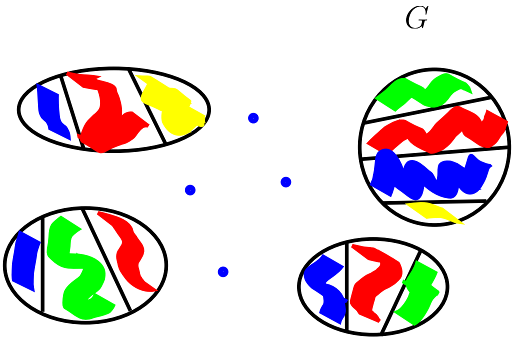

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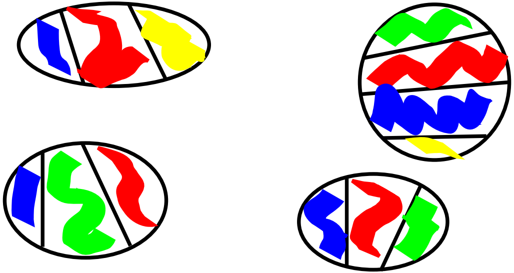

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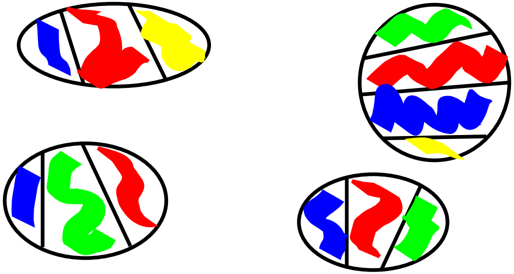

Observation: This yields a decomposition algorithm working in time $f(d) \cdot n^2$ on graphs of treedepth at most d.

Graph isomorphism for bounded treedepth

Theorem. [Bouland, Dawar, Kopczyński, 2012]

Graph isomorphism can be solved on graphs of treedepth at most d in time $f(d) \cdot n^3 \cdot \log n$.

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Remark: The running time can be further improved to $f(d) \cdot n \cdot \log^2 n$.

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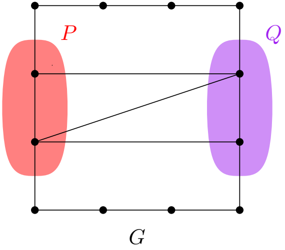

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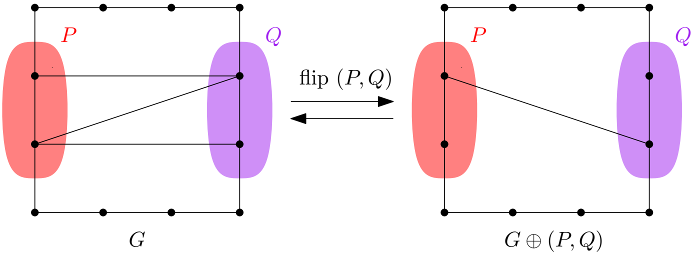

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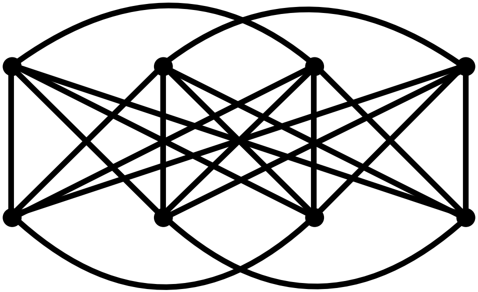

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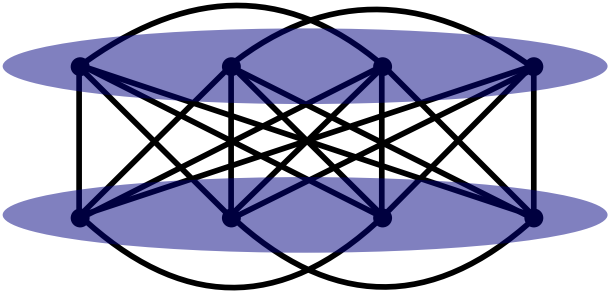

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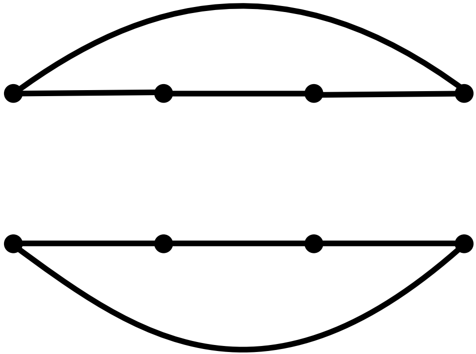

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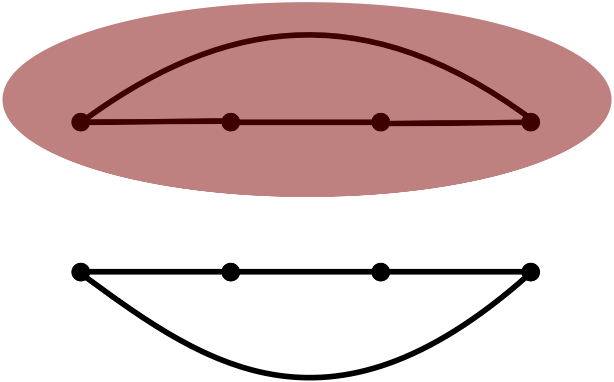

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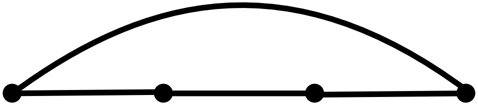

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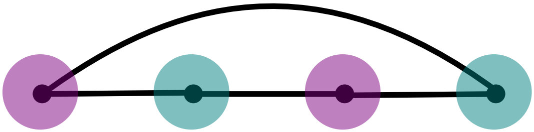

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Example play of the shrubdepth game:

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1. We found canonical moves for Splitter in the treedepth game
2. to obtain canonical decompositions and a graph isomorphism algorithm for graphs of bounded treedepth.

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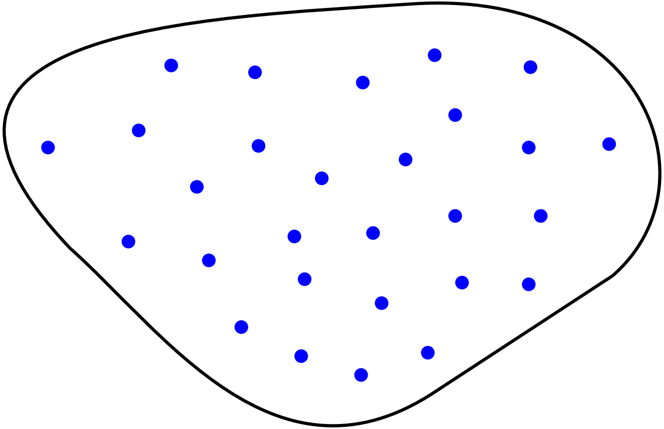
Defintion. [Ganian, Hliněný, Nešetřil, Obdržálek, Ossona de Mendez, 2017]

A graph G has *shrubdepth* at most d if Flipper can win the shrubdepth game on G in at most d rounds.

Finitary Substitute Lemma

$G \in \mathcal{C}$ $\phi(x_1, x_2; y_1, \ldots, y_k)$ ψ



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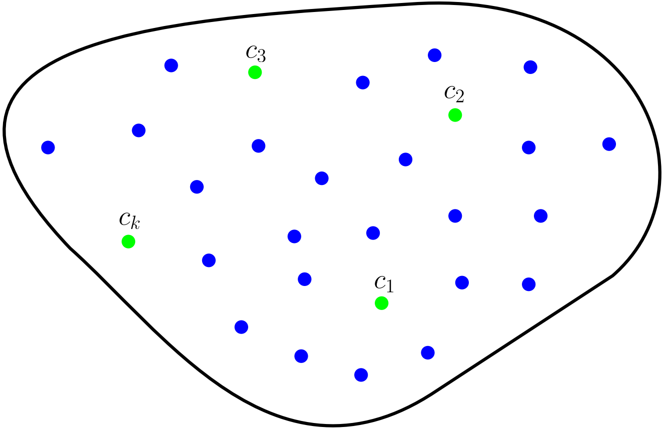

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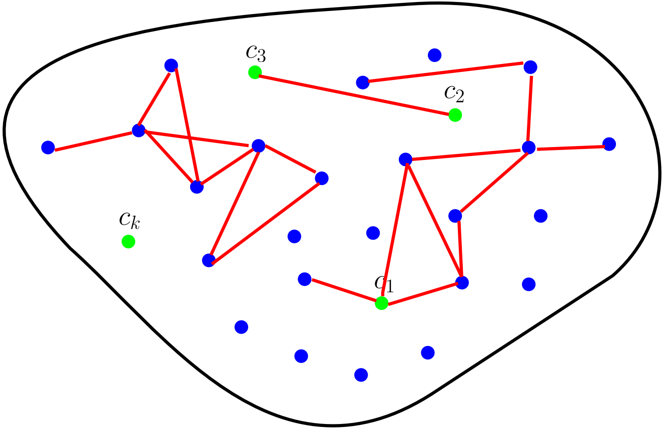

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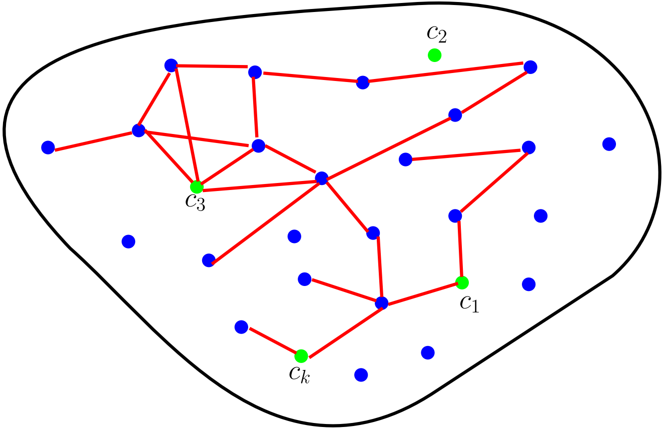
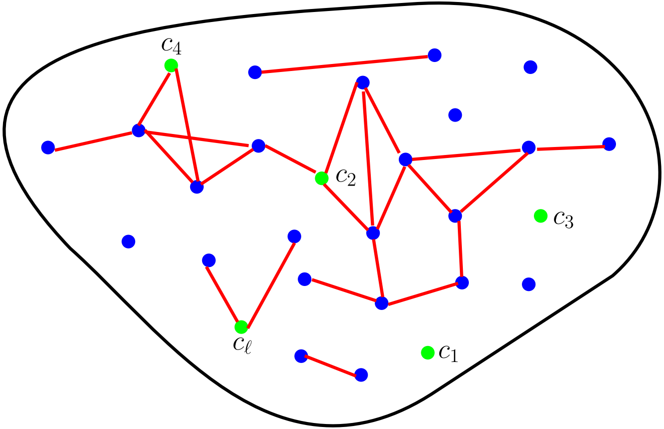


Finitary Substitute Lemma

$G \in \mathcal{C}$ $\phi'(x_1; x_2; y_1, \ldots, y_\ell)$ ψ



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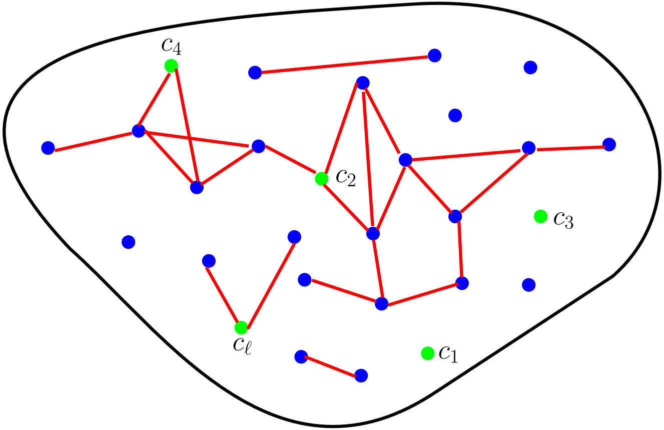

Proof uses a number of tools from stability theory [Shelah], most importantly properties of forking independence in stable theories.

Graph isomorphism on bounded shrubdepth

Theorem. [Ohlmann, Pilipczuk, Przybyszewski, Toruńczyk, 2023]

Graph isomorphism can be solved on graphs of shrubdepth at most d in time $f(d) \cdot n^2$.