Teoria współbieżności

Piotr Hofman
Theoretical aspects of concurrency

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Outline

1. How to specify properties of a system?
    - LTL.
    - CTL.
    - Bisimulation.
2. How to model a system?
    - Process algebra.
    - Petri nets.

Assessment methods and assessment criteria

Oral exam 0 up to 15 point

- 3 questions each for 0-5 points

In the end of the semester I will provide a list of questions that may appear on the exam.

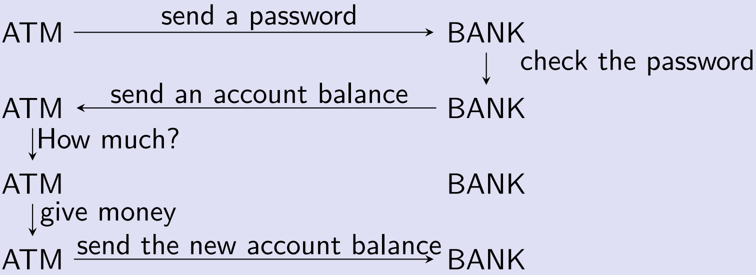
- $[0 - 8) \leftrightarrow 2$
- $[8 - 10) \leftrightarrow 3$
- $[10 - 11.5) \leftrightarrow 3+$
- $[11.5 - 13) \leftrightarrow 4$
- $[13 - 14) \leftrightarrow 4+$
- $[14 - 15] \leftrightarrow 5$

Basic problems with concurrent programs

Three basic threats in concurrent programming

Data corruption

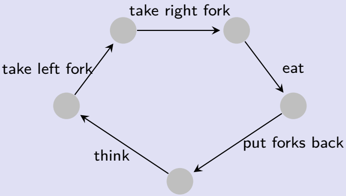
Consider a bank, an ATM, and a following protocol for withdrawing money:

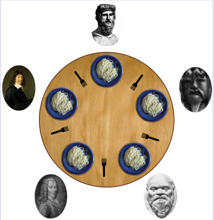


Three basic threats in concurrent programming

Deadlock

Consider philosophers working according a following schema:





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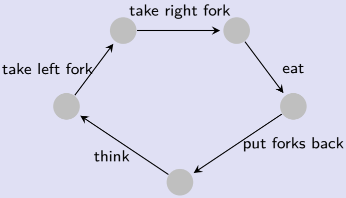

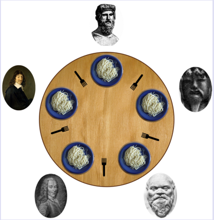

Solution

Priorities.

Three basic threats in concurrent programming

Starvation

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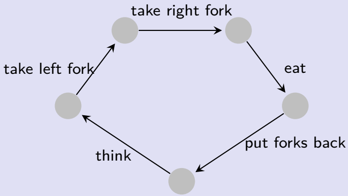

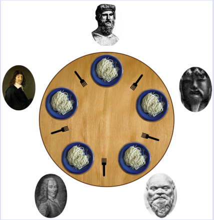

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Kripke structures

Let $\mathbb{AP}$ be a set of atomic propositions, i.e. boolean expressions over variables, constants and predicate symbols.

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Definition

A Kripke structure over $\mathbb{AP}$ is a 4-tuple $M = (S, I, R, L)$ consisting of:

1. a finite set of states S,
2. a set of initial states $I \subseteq S$,
3. a transition relation $R \subseteq S \times S$ such that R is left-total, i.e., $\forall_{s \in S} \exists_{s' \in S}$ such that $(s, s') \in R$,
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Definition

By a *run* we mean a sequence of states interleaved with transitions, $s_1, t_1, s_2, t_2, s_3 \ldots$ such that $(s_i, s_{i+1}) = t_i$.

Runs

Definition

For a given run we define a *trace* as follows:

$$\mathbb{TR}(s_1, t_1, s_2, t_2, s_3 \ldots) = L(s_1), L(s_2), L(s_3) \ldots .$$

A set of traces of all possible infinite runs starting in I (one of initial states) of a given Kripke structure S is called *Traces* of S. We denoted it $\mathbb{TR}(S)$.

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- Almost! It is under the assumption that model is correct and precise enough.

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Exercises

$5 \to 3$ philosophers

- What are the predicates?
- How the Kripke structure looks like?
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How to specify the above properties?

Automaton

Traces are languages, so we can try specify properties with automata.
Let Σ be a set of letters (a finite alphabet).

Definition

Automaton is an ordered 5-tuple $A = (S, I, F, R, L)$ where:

1. S is a finite set of states,
2. I is a set of initial states, $I \subseteq S$,
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Definition

A language of an automaton A is a set of words $\subseteq \Sigma^*$ such that they can be read along the paths from an initial state to a final state.

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Problem

Traces are infinite words and words accepted by a non-deterministic automaton are finite.

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Definition

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7. Number of b is finite and number of a between any two b is not divisible by 3.

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Büchi languages are closed under:

1. union,
2. intersection,
3. complement (We will not do this)
4. determinisation does not work (we need to extend the model).

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So we can use intersection and test for non-emptiness.

(LTL) Linear temporal logic.

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8. It should allow to say that some property holds after something happens.

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Definition

An *LTL* formula ϕ is generated according a following rules:

$$\phi \rightarrow true | p_i \in \mathbb{AP} | \phi_1 \wedge \phi_2 | \neg \phi_1 | X \phi_1 | \phi_1 U \phi_2$$

Semantics of LTL

$p_1 \rightarrow p_2 \rightarrow p_1 p_2 \rightarrow p_2 \rightarrow p_3 \rightarrow p_2 \rightarrow p_1 p_3 \rightarrow \cdots$

- $true \iff true$.
- $p_1 \iff p_1$ holds at position 0.
- $p_2 \iff p_2$ holds at position 0.
- $p_1 \wedge p_2 \iff p_1$ and p_2 holds at position 0.
- $\neg p_2 \iff p_2$ does not hold at position 0.
- $Xp_2 \iff p_2$ holds at position 1.
- $XX(p_1 \wedge p_2) \iff p_2$ and p_1 holds at position 2.
- $\neg(\neg p_1 \wedge \neg p_2) U p_3$
  $\iff$ there is $0 \leq j$ such that p_3 holds at position j and $\neg(\neg p_1 \wedge \neg p_2)$ holds for all $0 \leq i < j$.

Exercise

How to express:

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3. If p_2 holds at the state with index 2 then $p_3 \vee p_1$ holds in the state with index 4.
4. If in some state p_1 is satisfied then in the future p_2 has to be satisfied.

How to verify LTL formula?

Lemma

Let *words*(ϕ) denotes a set of words that satisfy the LTL formula ϕ. For a given LTL formula ϕ one can construct an exponential size Büchi automaton B recognising exactly the same set of words, i.e.

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1. Build an automaton A for the Kripke structure.
2. Build an automaton B for ϕ an LTL formula,
   or build an automaton B for $\neg\phi$.
3. Check non-emptiness of $\mathbb{L}(A) \cap \mathbb{L}(B)$.

Bibliography

Units from 3 to 8 from (ordered by date)
https://www.youtube.com/channel/UCUXDMaaobC01He1HBiFZnPQ/videos
There are a lot of videos first one is "A problem in concurrency".